Master

E4.29
C1.1
ISE4.55

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE PART IV: MEng and ACGI

OPTIMIZATION

Wednesday, 27 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A. Astolfi
Second Marker(s) : J.C. Allwright

[4.29]

Special instructions for invigilators: None

Information for candidates:

- ∇f denotes the gradient of the function f. Note that ∇f is a column vector.
- $\nabla^2 f$ denotes the Hessian matrix of the function f. Note that this is a symmetric square matrix.

[4.29]

1. Consider the function

$$f(x) = x_1^2 + x_1 x_2 + (x_1 - x_2)^4.$$

(a) Compute all stationary points of the function.
(Hint: obtain first $(x_1 - x_2)^3$ in terms of x_1 from the necessary conditions of optimality.) [6]

(b) Using second order sufficient conditions, *classify* the stationary points determined in part (a), *i.e.* say which is a local minimum, or a local maximum, or a saddle point. [6]

(c) Consider the point $p = (0, 0)$ and the direction $d = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Using the definition of a descent direction, show that d is a descent direction for f at p. [4]

(d) Perform an exact line search along the direction $d = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ starting at $p = (0, 0)$. Show that the point obtained as a result of the line search procedure is a local minimum of the function f. [4]

[4.29]

2. Consider the problem of minimizing the function

$$f(x) = x_1^2 + 2x_2^2 + 4x_1 + 4x_2.$$

(a) Compute the stationary points of the function. [2]

(b) Consider the minimization of the function f using the gradient algorithm. Express analytically the form of the generic iteration, *i.e.*

$$p_{k+1} = p_k - \alpha \nabla f$$

(where $p_i = [x_1^i, x_2^i]'$). [4]

(c) Compute three steps of the gradient algorithm with exact line search from the initial point $p_0 = [0, 0]'$, using the fact that, for this p_0 the exact line search parameter α is equal to $1/3$ **for all *k***. Check that $\alpha = 1/3$ for the first iteration. [8]

(d) Exploit the results of part (c) to show that the gradient iteration with exact line search for $p_0 = [0, 0]'$ gives

$$x_1^{k+1} = \frac{1}{3}x_1^k - \frac{4}{3}$$

$$x_2^{k+1} = -\frac{1}{3}x_2^k - \frac{4}{3},$$

and hence show that

$$(x_1^{k+1} + 2) = \frac{1}{3}(x_1^k + 2)$$

$$(x_2^{k+1} + 1) = -\frac{1}{3}(x_2^k + 1).$$

Hence, deduce that the sequence $\{p_k\}$ can be written as

$$p_{k+1} = \begin{bmatrix} \frac{2}{3^{k+1}} - 2 \\ \left(-\frac{1}{3}\right)^{k+1} - 1 \end{bmatrix}.$$

Show that the sequence $\{p_k\}$ converges to the stationary point determined in part (a). [6]

[4.29]

3. Consider the minimization problem

$$\begin{cases} \min_{x_1,x_2} -x_1 x_2 \\ 0 \le x_1 + x_2 \le 2 \\ -2 \le x_1 - x_2 \le 2 \end{cases}$$

(a) Sketch in the (x_1, x_2)-plane the level surfaces of the function to be minimized and the admissible set. Hence show that all points in the admissible set are regular points for the constraints. [6]

(b) Using only graphical considerations, determine the global solution of the considered problem. [4]

(c) State first order necessary conditions of optimality for such a constrained optimization problem. Show that the point determined in part (b) satisfies first order necessary conditions of optimality, for some selection of the multiplier ρ. [4]

(d) Show that the point determined in part (b) satisfies second order sufficient conditions of optimality for such a constrained optimization problem. [6]

[4.29]

4. Consider the optimization problem

$$\begin{cases} \min_{x_1,x_2,x_3} Tx_1^2 + Tx_2^2 + x_3^2 \\ x_1 + x_2 + x_3 - 1 = 0. \end{cases}$$

(a) Transform this minimization problem into an unconstrained minimization problem by solving the constraint equation for x_3 and substituting the solution into the objective function. [4]

(b) Assume $T > 0$. Consider the unconstrained minimization problem determined in part (a). Find (the unique) candidate optimal solution and show that this is indeed a local minimizer. [6]

(c) Assume $T > 0$. Exploit the results in parts (a) and (b) to determine the solution of the constrained optimization problem. [2]

(d) Assume $T > 0$. Consider the so-called l_1 penalty function

$$f_p = Tx_1^2 + Tx_2^2 + x_3^2 + \frac{|x_1 + x_2 + x_3 - 1|}{\epsilon}$$

with $\epsilon > 0$ and sufficiently small. Show that the unique stationary point of f_p coincides with the optimal solution determined in part (c).
(Hint: recall that that $\frac{d|x|}{dx} = \text{sign}(x)$ and that $\text{sign}(0) \in [-1, 1]$. Moreover, use the fact that the stationary points of f_p do not depend upon the parameter ϵ.) [8]

[4.29]

5. Consider the optimization problem

$$\begin{cases} \min\limits_{x_1,x_2} x_1 x_2 \\ x_1^2 + x_2^2 - 1 \leq 0. \end{cases}$$

(a) State first order necessary conditions of optimality for such a constrained optimization problem. [4]

(b) Using the conditions derived in part (a), compute candidate optimal solutions. [6]

(c) This constrained optimization problem can be transformed into an unconstrained optimization problem by defining the so-called barrier function

$$B_\epsilon(x) = x_1 x_2 + \frac{\epsilon}{1 - x_1^2 - x_2^2},$$

with $\epsilon > 0$, and considering the unconstrained minimization of $B_\epsilon(x)$. Determine the stationary points x_ϵ of $B_\epsilon(x)$.
(Hint: show that all stationary points $\bar{x} = (\bar{x}_1, \bar{x}_2)$ are such that $\bar{x}_1 = -\bar{x}_2$, and then note that the solutions of the equation

$$x - \frac{2\epsilon x}{(2x^2 - 1)^2} = 0$$

are $x = 0$ and $x = \pm\frac{\sqrt{2 \pm 2\sqrt{2\epsilon}}}{2}$.)
Discuss the feasibility of the obtained stationary points x_ϵ. Compute $\lim_{\epsilon \to 0} x_\epsilon$ and compare this result with the results obtained in parts (a) and (b). [6]

(d) Discuss the advantages and disadvantages of the proposed barrier function method in comparison with the sequential penalty functions method presented in the lectures. [4]

[4.29]

6. Consider the function

$$f = x_1^4 - x_1 x_2 + x_2^4$$

and the problem of finding a global minimum for f.

(a) Briefly describe the Branin method for global minimization. [4]

(b) Write the formulae for the so-called Branin system,

$$\dot{x} = -[\nabla^2 f]^{-1} \nabla f,$$

for the particular function f specified above. [2]

(c) Compute the equilibria of the Branin system determined in part (b). Show that these equilibria coincide with the stationary points of the function f. Show that f is radially unbounded. Hence determine the global minimum of f. [4]

(d) Consider the linearization of the Branin system, computed in part (b), around its equilibrium at $x = 0$. Show that this linearized system has two eigenvalues equal to -1, hence deduce that the point $x = 0$ is locally attractive. [4]

(e) Write now the formulae for the modified Branin system

$$\dot{x} = -\det(\nabla^2 f)[\nabla^2 f]^{-1} \nabla f,$$

for the function f above. Consider the linearization of the modified Branin system at the zero and show that this equilibrium point is unstable. [4]

(f) Give reasons for the modified Branin method being preferable to the Branin method when determining a global minimum for the function f above. [2]

MASTER

Optimisation - Model answers 2005

(Note to external examiners: all questions involve mostly applications of standard methods and concepts to unseen examples.)

Question 1

(a) The stationary points of the function f are computed by solving the equation

$$0 = \nabla f = \begin{bmatrix} 2x_1 + x_2 + 4(x_1 - x_2)^3 \\ x_1 - 4(x_1 - x_2)^3 \end{bmatrix},$$

yielding

$$P_1 = (0,0) \qquad P_2 = (-\frac{1}{16}, \frac{3}{16}) \qquad P_3 = (\frac{1}{16}, -\frac{3}{16}).$$

(b) Note that

$$\nabla^2 f = \begin{bmatrix} 2 + 12(x_1 - x_2)^2 & 1 - 12(x_1 - x_2)^2 \\ 1 - 12(x_1 - x_2)^2 & 12(x_1 - x_2)^2 \end{bmatrix}.$$

Thus

$$\nabla^2 f(P_1) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

which is an indefinite matrix,

$$\nabla^2 f(P_2) = \nabla^2 f(P_3) = \begin{bmatrix} 11/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

which is a positive definite matrix. As a result, P_1 is a saddle point, and P_2 and P_3 are local minima.

(c) By definition, a direction d is a descent direction for f at p if there exists $\delta > 0$ such that

$$f(p + \lambda d) < f(p),$$

for all $\lambda \in (0, \delta)$. Consider now the given direction and note that $f(p) = 0$ and that

$$f(p + \lambda d) = -2\lambda^2 + 256\lambda^4.$$

Hence, $f(p + \lambda d) < f(p)$ for all $\lambda > 0$ and sufficiently small. (Note that $\nabla f(p)'d = 0$, hence this condition cannot be used to decide if d is a descent direction.)

(d) To perform an exact line search, along the direction d starting from $p = (0,0)$, we need to find the minimum of the function

$$\phi(\lambda) = f(p + \lambda d) - f(p) = -2\lambda^2 + 256\lambda^4$$

for $\lambda > 0$. Note that

$$\frac{d\phi}{d\lambda} = -4\lambda + 1024\lambda^3,$$

hence the minimum is achieved for $\lambda = 1/16$. The resulting point is $(1/16, -3/16)$ and this coincides with one of the local minima determined in part (a).

Question 2

(a) The stationary points of the function f are computed by solving the equation

$$0 = \nabla f = \begin{bmatrix} 2x_1 + 4 \\ 4x_2 + 4 \end{bmatrix},$$

yielding the stationary point

$$p^\star = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

(b) The generic iteration of the gradient algorithm for the considered function f is

$$x_1^{k+1} = x_1^k - \alpha(2x_1^k + 4) \qquad x_2^{k+1} = x_2^k - \alpha(4x_2^k + 4).$$

(c) Setting $(x_1^0, x_2^0) = (0,0)$ one has $(x_1^1, x_2^1) = (-4\alpha, -4\alpha)$ and

$$f(-4\alpha, -4\alpha) - f(0,0) = 48\alpha^2 - 32\alpha.$$

Minimizing this function yields $\alpha = 1/3$ (as stated). Therefore, $(x_1^1, x_2^1) = (-4/3, -4/3)$. Repeating the same considerations, and setting always $\alpha = 1/3$, one has

$$(x_1^2, x_2^2) = (-16/9, -8/9)$$

and

$$(x_1^3, x_2^3) = (-52/27, -28/27).$$

(d) Setting $\alpha = 1/3$ in the gradient iteration yields

$$x_1^{k+1} = \frac{1}{3}x_1^k - \frac{4}{3} \qquad x_2^{k+1} = -\frac{1}{3}x_2^k - \frac{4}{3}$$

and this can be also written as

$$(x_1^{k+1} + 2) = \frac{1}{3}(x_1^k + 2) \qquad (x_2^{k+1} + 1) = -\frac{1}{3}(x_2^k + 1).$$

As a result

$$(x_1^{k+1} + 2) = \left(\frac{1}{3}\right)^k (x_1^0 + 2) \qquad (x_2^{k+1} + 1) = \left(-\frac{1}{3}\right)^k (x_2^0 + 1),$$

or, equivalently,

$$x_1^{k+1} = 2\left(\frac{1}{3}\right)^k - 2 \qquad x_2^{k+1} = \left(-\frac{1}{3}\right)^k - 1.$$

Finally, as $k \to \infty$ $x_1^k \to -2$ and $x_2^k \to -1$, *i.e.* the sequence converges to the stationary point determined in part (a).

Question 3

(a) The admissible set is the shaded area in Figure 1. The arrows denote the gradient of the constraints on the boundary of the admissible set. As can be seen, these vectors are always independent, therefore all points are regular points for the constraints. The dashed lines represent level lines of the function f.

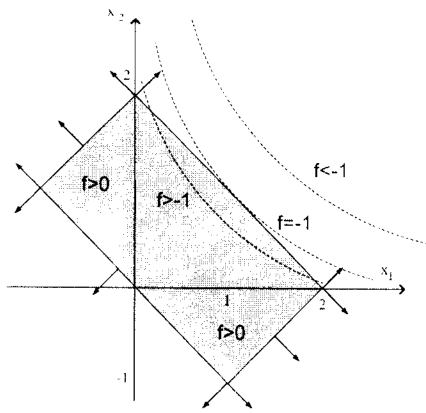

Figure 1: Admissible set and level lines of the function f.

(b) From Figure 1 it can be seen that the minimum is achieved when the level line of the function f is tangent to the admissible set, *i.e.* at the point $p = (1,1)$.

(c) Consider the Lagrangian

$$L = -x_1x_2 + \rho_1(-x_1 - x_2) + \rho_2(x_1 + x_2 - 2) + \rho_3(-2 - x_1 + x_2) + \rho_4(x_1 - x_2 - 2).$$

The first order sufficient conditions of optimality are

$$\nabla_x L = 0 = \begin{bmatrix} -x_2 - \rho_1 + \rho_2 - \rho_3 + \rho_4 \\ -x_1 - \rho_1 + \rho_2 + \rho_3 - \rho_4 \end{bmatrix}$$

$$-x_1 - x_2 \leq 0 \qquad x_1 + x_2 - 2 \leq 0 \qquad -x_1 - x_2 - 2 \leq 0 \qquad x_1 - x_2 - 2 \leq 0,$$

$$\rho_1 \geq 0 \qquad \rho_2 \geq 0 \qquad \rho_3 \geq 0 \qquad \rho_4 \geq 0$$

$$\rho_1(-x_1 - x_2) = 0 \qquad \rho_2(x_1 + x_2 - 2) = 0 \qquad \rho_3(-2 - x_1 + x_2) = 0 \qquad \rho_4(x_1 - x_2 - 2) = 0.$$

Setting $(x_1, x_2) = (1,1)$ and selecting $\rho_1 = 0$, $\rho_3 = 0$, $\rho_4 = 0$ satisfies all the above equations. Hence, the point $(x_1, x_2) = (1,1)$, together with the given multipliers, satisfies first order necessary conditions of optimality.

(d) To check second order sufficient conditions note that, for $(x_1, x_2) = (1,1)$, the only active constraint is $x_1 + x_2 - 2 \leq 0$. Therefore we need to check positivity of $s'\nabla^2_{xx}Ls$ for $s = [s_1\ s_2]'$ such that $[1\ 1]s = 0$. This means $s_1 + s_2 = 0$, hence, solving for s_2, one has

$$s'\nabla^2_{xx}Ls = \begin{bmatrix} s_1 & -s_1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ -s_1 \end{bmatrix} = 2s_1^2 > 0$$

for $s_1 \neq 0$. As a result, the point obtained from graphical considerations in part (b) is indeed a local minimizer for the considered problem.

Question 4

(a) Solving the constrain equation for x_3 yields $x_3 = 1 - x_1 - x_2$. This is replaced in the function to minimize, hence resulting in the unconstrained minimization problem

$$\min_{x_1,x_2} \tilde{f}$$

with

$$\tilde{f} = Tx_1^2 + Tx_2^2 + (1 - x_1 - x_2)^2.$$

(b) To determine candidate optimal solution consider the equations

$$0 = \nabla\tilde{f} = \begin{bmatrix} 2Tx_1 + 2x_1 + 2x_2 - 2 \\ 2Tx_2 + 2x_1 + 2x_2 - 2. \end{bmatrix}$$

These have the unique solution

$$x_1^\star = \frac{1}{T+2} \qquad x_2^\star = \frac{1}{T+2}.$$

Note now that

$$\nabla^2\tilde{f} = \begin{bmatrix} 2T+2 & 2 \\ 2 & 2T+2 \end{bmatrix}$$

and this is positive definite for $T > 0$. Hence, the obtained stationary point is a local minimizer for $\tilde{f}$.

(c) To obtain a solution of the original problem it is enough to compute

$$x_3^\star = 1 - x_1^\star - x_2^\star = \frac{T}{T+2}.$$

(d) To compute the stationary points of f_p (recall that $\frac{d|x|}{dx} = \text{sign}(x)$) consider the equations

$$0 = \nabla f_p = \begin{bmatrix} 2Tx_1 + \frac{\text{sign}(x_1+x_2+x_3-1)}{\epsilon} \\ 2Tx_2 + \frac{\text{sign}(x_1+x_2+x_3-1)}{\epsilon} \\ 2x_3 + \frac{\text{sign}(x_1+x_2+x_3-1)}{\epsilon} \end{bmatrix}.$$

These can be rewritten as

$$2Tx_1 = 2Tx_2 = 2x_3 = -\frac{\text{sign}(x_1 + x_2 + x_3 - 1)}{\epsilon}$$

yielding

$$x_1 = x_3/T \qquad x_2 = x_3/T.$$

Replacing this in the last equation yields

$$2x_3 = -\frac{\text{sign}(x_3/T + x_3/T + x_3 - 1)}{\epsilon}.$$

Note now that the solution of this equation may be independent of ϵ only if $x_3/T + x_3/T + x_3 - 1 = 0$, implying $x_3 = x_3^\star$. Finally, this implies that $x_1 = x_1^\star$ and $x_2 = x_2^\star$, *i.e.* the unique stationary point of f_p coincides with the optimal solution obtained in part (c).

Question 5

(a) Let

$$L = x_1x_2 + \rho(x_1^2 + x_2^2 - 1).$$

The first order sufficient conditions of optimality are

$$0 = \nabla_x L = \begin{bmatrix} x_2 + 2\rho x_1 \\ x_1 + 2\rho x_2 \end{bmatrix}$$

$$x_1^2 + x_2^2 - 1 \le 0 \qquad \rho \ge 0$$

$$(x_1^2 + x_2^2 - 1)\rho = 0.$$

(b) From the first two equations we have that if $\rho \neq 1/2$ then $x_1 = x_2 = 0$. If $\rho = 1/2$ then $x_1 + x_2 = 0$ and from the last equation $x_1^2 + x_2^2 - 1 = 0$. As a result $x_1 = \pm\frac{1}{\sqrt{2}}$ and $x_2 = \mp\frac{1}{\sqrt{2}}$. In conclusion we have three candidate solutions

$$P_1 = (0,0) \qquad P_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \qquad P_3 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

(c) To determine the stationary points of B_ϵ, consider the equation

$$0 = \nabla_x B_\epsilon = \begin{bmatrix} x_2 + 2\epsilon\frac{x_1}{(1-x_1^2-x_2^2)^2} \\ x_1 + 2\epsilon\frac{x_2}{(1-x_1^2-x_2^2)^2} \end{bmatrix}.$$

From these we obtain

$$\begin{bmatrix} \frac{x_2}{x_1} = -2\epsilon\frac{1}{(1-x_1^2-x_2^2)^2} \\ \frac{x_1}{x_2} = -2\epsilon\frac{1}{(1-x_1^2-x_2^2)^2} \end{bmatrix}$$

hence $x_1/x_2 < 0$ and $x_2/x_1 = x_1/x_2$. As a result $x_1 = -x_2$. Replacing in the second equation yields

$$x_1 = 2\epsilon\frac{x_1}{(1-2x_1^2)^2}$$

hence we obtain five candidate solutions, namely

$$P_a = (0,0) \qquad P_b = (\frac{\sqrt{2+2\sqrt{2\epsilon}}}{2}, -\frac{\sqrt{2+2\sqrt{2\epsilon}}}{2}) \qquad P_c = (-\frac{\sqrt{2+2\sqrt{2\epsilon}}}{2}, \frac{\sqrt{2+2\sqrt{2\epsilon}}}{2})$$

$$P_d = (\frac{\sqrt{2-2\sqrt{2\epsilon}}}{2}, -\frac{\sqrt{2-2\sqrt{2\epsilon}}}{2}) \qquad P_e = (-\frac{\sqrt{2-2\sqrt{2\epsilon}}}{2}, \frac{\sqrt{2-2\sqrt{2\epsilon}}}{2}).$$

Note that P_a, P_d and P_e are feasible, whereas P_b and P_c are not feasible. Finally $P_a = P_1$,

$$\lim_{\epsilon\to 0} P_b = \lim_{\epsilon\to 0} P_d = P_2$$

and

$$\lim_{\epsilon\to 0} P_c = \lim_{\epsilon\to 0} P_e = P_3$$

(d) The proposed method is preferable to the sequential penalty function method because it provides feasible solutions also for $\epsilon > 0$. However, the function B_ϵ is not defined on all $I\!R^2$, hence it may be difficult to perform a numerical minimization.

Question 6

(a) (From the lecture notes)

The Branin method can be described as follows. Consider the function f and assume ∇f is continuous. Fix x_0 and consider the differential equations

$$\frac{d}{dt}\nabla f(x(t)) = \pm\nabla f(x(t)) \qquad x(0) = x_0. \tag{1}$$

The solutions $x(t)$ of such differential equations are such that

$$\nabla f(x(t)) = \nabla f(x_0)e^{\pm t}.$$

Using this fact, it is possible to describe Branin algorithm.

Step 0. Given x_0.

Step 1. Compute the solution $x(t)$ of the differential equation

$$\frac{d}{dt}\nabla f(x(t)) = -\nabla f(x(t))$$

with $x(0) = x_0$.

Step 2. $x^\star = \lim_{t\to\infty} x(t)$ is a stationary point of f, in fact $\lim_{t\to\infty} \nabla f(x(t)) = 0$.

Step 3. Consider a perturbation of the point $x^\star$, *i.e.* the point $\tilde{x} = x^\star + \epsilon$ and compute the solution $x(t)$ of the differential equation

$$\frac{d}{dt}\nabla f(x(t)) = \nabla f(x(t)).$$

Along this trajectory the gradient $\nabla f(x(t))$ increases, hence the trajectory escapes from the *region of attraction* of x_0.

Step 4. Fix $\bar{t} > 0$ and assume that $x(\bar{t})$ is sufficiently away from x_0. Set $x_0 = x(\bar{t})$ and go to **Step 1**.

Note that, if the perturbation ϵ and the time $\bar{t}$ are properly selected, at each iteration the algorithm generates a new stationary point of the function f.

If $\nabla^2 f$ is continuous then the differential equations (1) can be written as

$$\dot{x}(t) = \pm\left[\nabla^2 f(x(t))\right]^{-1}\nabla f(x(t)).$$

Therefore Branin method is a continuous equivalent of Newton method. Note however that, as $\nabla^2 f(x(t))$ may become singular, the above equation may be meaningless.

(b) The Branin system is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{1}{144x_1^2x_2^2 - 1}\begin{bmatrix} -48x_2^2x_1^3 - 8x_2^3 - x_1 \\ 8x_1^3 + x_2 - 48x_1^2x_2^3 \end{bmatrix}.$$

(c) The equilibria of the Branin system are $P_1 = (0,0)$, $P_2 = (1/2, 1/2)$ and $P_3 = (-1/2, -1/2)$. Note now that these are also such that $\nabla f(P_i) = 0$, for $i = 1,2,3$. Hence, the equilibria of Branin system coincide with the stationary points of f. Note now that $\lim_{\|x\|\to\infty} f(x) = +\infty$, hence f is radially unbounded. Moreover $f(P_1) = 0$ and $f(P_2) = f(P_3) = -1/8$. Hence the global minimum of f is $-1/8$ and there are two global minimizers, P_2 and P_3.

(d) The linearization of the Branin system around the point P_1 is described by

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x,$$

and this shows that the point P_1 is locally attractive, and that the linearized system has two eigenvalues equal to -1.

(e) The modified Branin system is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -48x_2^2x_1^3 - 8x_2^3 - x_1 \\ 8x_1^3 + x_2 - 48x_1^2x_2^3 \end{bmatrix}.$$

Its linearization around P_1 is described by

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x,$$

and this shows that the point P_1 is an unstable equilibrium for the modified Branin system.

(f) The modified Branin system has the following advantages:

- the differential equations are defined for all x;
- the point P_1, which is a local maximum, is unstable therefore the trajectories of the system will not be *attraced* by P_1.