IMPERIAL COLLEGE LONDON

E4.27
C2.3
ISE4.41

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

MSc and EEE/ISE PART IV: M.Eng. and ACGI

SYSTEM IDENTIFICATION

Tuesday, 29 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Weiss
Second Marker(s) : J.C. Allwright

Special information for invigilators: none

Information for candidates:

$$C(\tau) = E[(u(t)-\mu)(u(t+\tau)-\mu)]$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \qquad S_{yy} = |G|^2 S_{uu} \qquad Z_L = sL \qquad Z_c = \frac{1}{Cs}$$

$$\Phi^{\#} = (\Phi^*\Phi)^{-1}\Phi^* \qquad P = \Phi\Phi^{\#} \qquad S = \frac{1}{N-\rho}\|y - \Phi\widehat{\theta}\|^2$$

$$A^d = e^{Ah} \qquad B^d = \left(e^{Ah} - I\right)A^{-1}B \qquad G^d(z) \approx G\left(\tfrac{2}{h}\,\tfrac{z-1}{z+1}\right) \qquad G(s) \approx G^d\left(\tfrac{1+sh/2}{1-sh/2}\right)$$

$$C_k^{uu} g_0 + C_{k-1}^{uu} g_1 + C_{k-2}^{uu} g_2 + \ldots = C_k^{uy}$$

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$E(X\cdot Y) = E(X)\cdot E(Y) + \mathrm{Cov}(X,Y)$$

$$\mathrm{Cov}(TX) = T\mathrm{Cov}(X)T^*$$

$$\widehat{v}(z) = \sum_{k=0}^{\infty} v_k z^{-k}$$

$$[(\Delta v)_k = v_{k+1}] \quad \Rightarrow \quad \Delta v\,(z) = z[\widehat{v}(z) - v_0]$$

$$[u_k = k v_k] \quad \Rightarrow \quad \widehat{u}(z) = -z\frac{d}{dz}\widehat{v}(z)$$

$$[v_k = \sin k\nu] \quad \Rightarrow \quad \widehat{v}(z) = \frac{z\sin\nu}{(z-e^{i\nu})(z-e^{-i\nu})}$$

$$\left[v_k = \rho^k\right] \quad \Rightarrow \quad \widehat{v}(z) = \frac{z}{z-\rho}$$

$$\left[v_k = \frac{1}{\rho}k\rho^k\right] \quad \Rightarrow \quad \widehat{v}(z) = \frac{z}{(z-\rho)^2}$$

$$y_k + a_1 y_{k-1} \ldots + a_n y_{k-n} = b_0 u_k + b_1 u_{k-1} \ldots + b_n u_{k-n} + e_k + c_1 e_{k-1} \ldots + c_n e_{k-n}$$

$$C(z) = 1 + c_1 z^{-1} \ldots + c_n z^{-n}$$

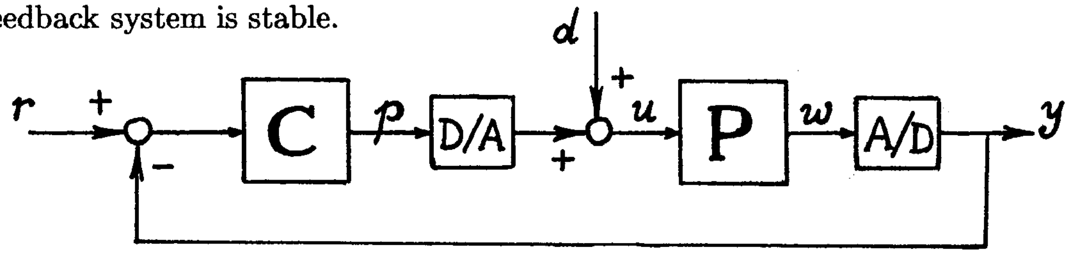
$$\hat{u}^F = C^{-1}\hat{u}, \qquad \hat{y}^F = C^{-1}\hat{y}$$

$$\overline{y_k} = (c_1 - a_1)y_{k-1}^F + (c_2 - a_2)y_{k-2}^F \ldots + (c_n - a_n)y_{k-n}^F + b_0 u_k^F + b_1 u_{k-1}^F \ldots + b_n u_{k-n}^F$$

1. Consider the feedback system shown in the figure below, where a continuous-time plant with transfer function **P** is controlled by a discrete-time controller with known clock period $h > 0$ and transfer function **C**. Here, r is a variable reference signal and d is a *constant* disturbance signal. The signals r, p and y are discrete-time, while d, u and w are continuous-time. As usual, the D/A converter is a zero order hold of period h, while the A/D converter is a sampler of period h. We have

$$\mathbf{P}(s) = \frac{c}{1+Ts}, \qquad \mathbf{C}(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} \ldots + c_9 z^{-9}.$$

The coefficients of **C** are known, while c, T and d are unknown and should be estimated. The true transfer function of the plant may be more complicated, but we would like to model it by the simple function given above. The whole feedback system is stable.



(a) Assuming that **P** is given by the simple formula above, compute (exactly) the tranfer function $\mathbf{P}^d$ of the discretized plant from p to y. For which values of c and T is $\mathbf{P}^d$ stable? [4]

(b) Suppose that the values r_k, y_k are available for $1 \leq k \leq 2000$. By defining new variables if necessary, find a model of the system of form $y_k = \varphi_k \theta + e_k$, where y_k and φ_k are known, θ is the vector of unknown parameters and e_k are the equation errors. [4]

(c) Describe a least squares based method for estimating c, T and d from the measurements of r_k and y_k, $1 \leq k \leq 2000$, using the model derived in part (b). [4]

(d) We denote by $\hat{\theta}$ the least squares estimate of θ for the model derived in part (b). Which of the three reference signals listed below will lead to the smallest covariance matrix Cov $\hat{\theta}$ (as measured by its norm)? Which will lead to the smallest value for $\widehat{\text{Var}}\, e_k$, the estimated variance of e_k? Give a brief reasoning.

$$\text{(i) } r_k = 1, \qquad \text{(ii) } r_k = \cos 0.2k,$$

$$\text{(iii) } r_k = \text{white noise with } E(r_k) = 0, \quad \text{Var } r_k = 1. \qquad [4]$$

(e) If c and T have been found, how can we approximate the transfer function from p to y by a FIR transfer function of order 10? [4]

2. Two balls are floating in a pool of still water, in fixed horizontal positions, at a distance of 6 m from each other. When a small vertical force acts on the first ball, it creates waves. After about 4 sec, the waves reach the second ball, causing it to oscillate vertically. If the first ball stops moving, then the oscillations die down quickly (in particular, there are no waves being reflected from the sides of the pool). We make measurements of the vertical displacement y of the second ball using a digital circuit, obtaining the measurements $y_k = y(kh)$, $k = 1, 2, 3, \ldots$, where $h = 40$ msec is the sampling period. The force acting on the first ball is constant on each sampling interval (i.e., between two multiples of h). We would like to estimate the discrete-time transfer function from the sampled force acting on the first ball to the sampled vertical displacement of the second ball. We would like to model this transfer function as follows:

$$\mathbf{G}^d(z) = z^{-m} \cdot \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

where the positive integer m and the real numbers b_0, b_1, b_2, a_1, a_2 should be determined. The true transfer function is much more complicated.

To find the impulse response, we start from rest at $t = 0$ and push the first ball down with a known force $F > 0$ for $t \in [0, h]$. For $t > h$ the force becomes zero. We have 300 relevant measurements of the vertical displacement, $y_1, y_2, \ldots y_{300}$. For $k > 300$ we have $y_k = 0$.

(a) Suggest a way for estimating a narrow range of suitable m. [4]

(b) Assuming that an m has been chosen, describe a least squares based method for estimating the remaining unknown parameters. [5]

(c) If the estimation procedure you described in your answer to part (b) is repeated (using the same data y_k) for all m in the narrow range determined as in part (a), how can we decide which value m is the best? [3]

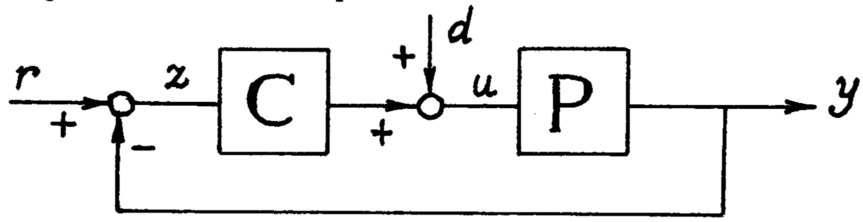
(d) Suppose that good estimates for m, b_0, b_1, b_2, a_1, a_2 have been found. How can we estimate the continuous-time transfer function from F to y, in the form

$$\mathbf{G}(s) = e^{-\tau s}\mathbf{G}_0(s),$$

where $\mathbf{G}_0$ is a proper rational transfer function of low order? [4]

(e) Suppose that a second similar experiment is made to measure the impulse response of the system. The new results are $\tilde{y}_1, \tilde{y}_2, \ldots \tilde{y}_{300}$, which are close (but not equal) to $y_1, y_2, \ldots y_{300}$. How can we use the new data to improve our model $\mathbf{G}^d$? [4]

3. For an unstable linear SISO plant with an unknown transfer function **P**, we want to design a stabilizing controller such that the output signal y should track any reference signal r which is sinusoidal with a frequency of 50 Hz. A random disturbance d also acts on the system. At frequencies higher than 2000 Hz the gain of the plant is practically zero. The feedback system is shown in the diagram below. We would like to make the influence of d on the tracking error z as small as possible.



To achieve tracking, we use a controller with the transfer function

$$\mathbf{C}(s) = k_1 + \frac{k_2 s}{s^2 + (100\pi)^2},$$

where k_1, k_2 are parameters to be tuned. It is known from experiments that for $k_1 = 1$ and $-50 \leq k_2 \leq 200$, this feedback system is stable.

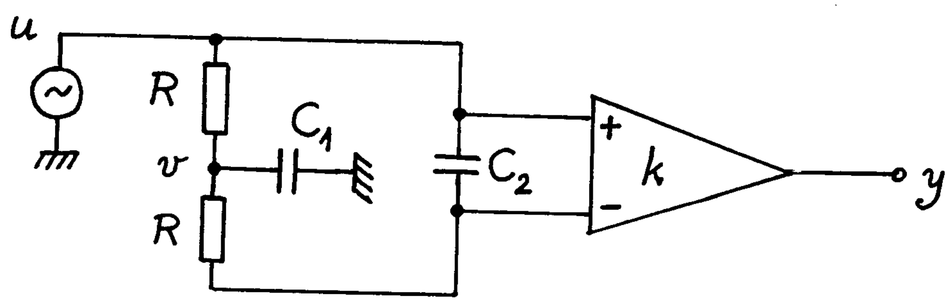
(a) Compute **S**, the transfer function from r to z, in terms of k_1, k_2 and **P**. If k_1 and k_2 are such that the system is stable, $d = 0$ and $r(t) = R\cos(\omega t)$, describe the behaviour of $z(t)$ for large $t > 0$. Comment in particular about the case when the frequency of r is 50 Hz. [5]

(b) In order to choose good values for k_1 and k_2, we would need an approximate Bode plot of **P**. Describe identification experiments which can provide us with the necessary data for the Bode plot. For these experiments, we are allowed to use the controller (if needed) and we can generate any bounded signal r. We cannot generate d, but we can set up the experiments in such a way that $d = 0$. Describe briefly the computations necessary to process the data from the identification experiments. [6]

(c) Assume that the feedback system is stable, r and d are independent stationary random signals with expectations $E(r) = E(d) = 0$ and known power spectral densities S_{rr} and S_{dd}. Is z a stationary random signal? Compute $E(z)$ and write the formulas needed for computing Var (z) (the power of z), in terms of **P**, **C**, S_{rr} and S_{dd} (do not do any computation). [6]

(d) If $k_1 = 1$, $k_2 = 100$, $r(t) = R\cos(100\pi t)$ for all $t \in \mathbb{R}$ and d is as in part (c), compute $E(y(t))$. Give a very brief reasoning. Hint: use your answer to part (a). [3]

4. In the model circuit shown below, $R = 1k\Omega$ while C_1, C_2 and the gain k of the differential amplifier are unknown positive quantities. No current is flowing to the inputs of the amplifier. We can choose the waveform of u and we can measure the output voltage y. The true circuit is more complicated than the model circuit shown, and hence we cannot expect a perfect match between its response and the response of the model, but we would like to get a close match in a certain frequency range.



(a) Choose state variables and construct a state space representation of the model circuit, of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, where x is the state and A, B, C and D are matrices. Is this (model) system stable? [6]

(b) Compute the transfer function $\mathbf{G}$ of the model circuit (from u to y), in terms of R, C_1 and C_2. Evaluate the gain of $\mathbf{G}$ for very low and for very high frequencies (i.e., for $\omega \to 0$ and for $\omega \to \infty$). [6]

(c) Suppose that by measurements that use sinusoidal u, we have obtained estimates for $\mathbf{G}$ at 50 angular frequencies $\omega_1, \ldots \omega_{50}$, in the frequency range of interest. Using these data, how could we estimate C_1, C_2 and k using a least squares based algorithm? Write down the formulas which give the estimated C_1, C_2 and k, taking care to define all the symbols that you use. Take care to make sure that the estimates for C_1, C_2 and k are real. [8]

5. A feedback system is obtained by interconnecting a discrete-time plant P with input u and output y described by

$$y_k - 0.9y_{k-1} = u_k - 0.8u_{k-1}$$

with a "differential" controller C with input y and output p described by

$$p_k = \delta(y_k - y_{k-1}),$$

where δ is an adjustable gain. The interconnection is such that $u = v - p$, where v is an external input signal.

(a) Draw a block diagram of the feedback system and compute the closed-loop transfer function $\mathbf{G}$ from v to y. [3]

(b) Check the stability of $\mathbf{G}$ for $\delta = -1, 0, 1$. [3]

(c) If δ is such that $\mathbf{G}$ is stable, compute the DC-gain of $\mathbf{G}$. Give a simple explanation why the DC-gain is independent of δ. [2]

(d) Assume $\delta = 1$. Explain why the step response of $\mathbf{G}$ is of the form

$$y_k = 2 + c_1(\lambda_1)^k + c_2(\lambda_2)^k \quad \text{for all} \quad k \geq 0,$$

where $|\lambda_1| < 1$, $|\lambda_2| < 1$. Explain briefly how the constants $c_1, \lambda_1, c_2, \lambda_2$ can be computed, but do not compute them. [5]

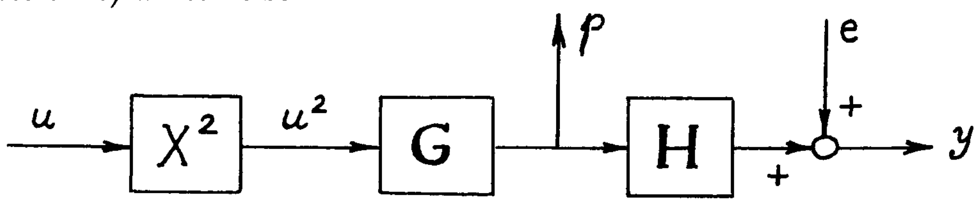
(e) Assume that the plant P has been obtained by a discrete-time identification procedure applied to a continuous-time LTI system, via sample and hold blocks (i.e., D/A and A/D converters) with a sampling frequency of 2 kHz. Make an estimate of the transfer function $\mathbf{P}^c$ of the continuous-time system, which should be valid for frequencies that are significantly lower than the sampling frequency. [2]

(f) Suppose that the feedback system is stable and the output measurements y_k of the system P are subject to measurement errors e_k, so that the controller C receives the corrupted measurement signal $\tilde{y}_k = y_k + e_k$, where e_k is normalized white noise (in particular, $E(e_k) = 0$ and $E(e_k^2) = 1$). Thus, the equation of C is now

$$p_k = \delta(\tilde{y}_k - \tilde{y}_{k-1}).$$

Given the corrupted measurements $\tilde{y}_0, \tilde{y}_1, \ldots \tilde{y}_{100}$ (the input signal v is also known), how can we compute an unbiased prediction of $\tilde{y}_{101}$? How large is the variance of the prediction error? [5]

6. We know from physical considerations that a nonlinear discrete-time system Σ is formed by the cascade connection of two stable LTI subsystems, and the input to the first linear subsystem is u^2, where u is the input signal of Σ. In the block diagram of Σ, shown below, the block marked X^2 is a static squaring block. The transfer functions of the LTI subsystems are denoted by **G** and **H**. The output y of Σ is obtained from the output of the second subsystem, but it is corrupted by the measurement noise e, which is (discrete-time) white noise.



Suppose that u is (discrete-time) white noise independent of e. The values u_k, p_k and y_k have been observed for $k = 0, 1, \ldots 10,000$. Based on these data, we would like to estimate the transfer functions **G** and **H**.

(a) Is u^2 stationary? Describe a method for estimating $E(u_k^2)$ and $C_\tau^{u^2u^2}$ (the autocorrelation function of u^2) for $\tau = 0, 1, \ldots 100$. [3]

(b) Describe a method for estimating the auto-correlation function C_τ^{pp} and the cross-correlation functions $C_\tau^{u^2p}$ and C_τ^{py} for $\tau = 0, 1, \ldots 100$. Explain very briefly how this problem is related to the concept of ergodicity. Is y ergodic? [3]

(c) Describe a method for estimating the terms $g_0, g_1, \cdots g_{100}$ in the impulse response of **G** from the results of parts (a) and (b). [3]

(d) Describe a method for estimating the terms $h_0, h_1 \ldots h_{100}$ in the impulse response of **H** from the results of part (b), and explain briefly how this method is derived from the properties of C_τ^{pp} and C_τ^{py}. [5]

(e) Having estimated $g_0, g_1, \cdots g_{100}$ from part (c), how can we build a FIR filter whose transfer function is a good approximation to **G**? Write the corresponding difference equation. [2]

(f) Express S^{yy}, the power spectral density of y, in terms of $E(u_k^2)$, $E(u_k^4)$, $E(e_k)$, $E(e_k^2)$, **G** and **H**. [4]

[END]