Estimation and Fault Detection 2003.

1 Solution

a) Solve the noiseless equation:

$$\ddot{x}_1 = 0$$

or $\dot{x}_2 = 0$

So
$$x_2(t) = x_2(s) \qquad \text{for } t > s$$
$$x_1(t) = x_1(s) + \int_s^t x_2(r)\,dr = x_1(s) + x_2(s)(t-s)$$

6 So
$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t-s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}$$

which gives the fundamental matrix. The noisy case is then an application of the variation-of-constants formula.

b) The best predictor conditional mean of $x(t)$ is the mean

$$\mathbb{E}\,x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$$

8 The covariance is just that of the integral noise:

$$\mathrm{Cov}(x(t)) = \int_0^t\int_0^t \begin{pmatrix} 1 & t-r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot(0\ 1)\begin{pmatrix} 1 & 0 \\ t-r' & 1 \end{pmatrix}\delta(r-r')\,dr\,dr'$$

$$= \int_0^t \begin{pmatrix} t-r \\ 1 \end{pmatrix}(t-r,\ 1)\,dr$$

$$= \int_0^t \begin{pmatrix} (t-r)^2 & t-r \\ t-r & 1 \end{pmatrix} dr = \begin{bmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{bmatrix}$$

c) $$E[x(t)\,|\,y_1] = E[x(t)] + \mathrm{Cov}(x(t), y_1)(\mathrm{Cov}\,y_1)^{-1}(y_1 - x_1(0))$$

7 But
$$\mathrm{Cov}(x(t)\,y_1) = \begin{matrix} \mathrm{Cov}(x_1) \\ \mathrm{Cov}(x_1, x_2) \end{matrix} = \begin{pmatrix} t^3/3 \\ t^2/2 \end{pmatrix}$$

$$\mathrm{Cov}(y_1) = \frac{t^3}{3} + Q$$

So $E[x(t)|y_1]$ has the form indicated with $K = \begin{pmatrix} \dfrac{1}{1 + \frac{3Q}{t^3}} \\ \dfrac{\cancel{t}}{\frac{2t}{3} + \frac{2Q}{t^2}} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $t \to \infty$.

As $t \to \infty$, the estimate 'forgets' the initial condition and relies solely on the single observation.

2 Solution Since all random variables are normal, we have

$$-\frac{1}{2}(x_0 - \hat{x}_{0|k})^T P_{0|k}^{-1} (x_0 - \hat{x}_{0|k}) \quad + \log p(y_k | y_{k-1} \cdots)$$

$$= -\frac{1}{2}(y_k - CA^k x_0)^T Q^{-1} (y_k - CA^k x_0)$$

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$$-\frac{1}{2}(x_0 - \hat{x}_{0|k-1})^T P_{0|k-1}^{-1} (x_0 - \hat{x}_{0|k-1})$$

Equating linear and quadratic terms in x_0 gives the two expressions for the information filter

b) If $x_{k+1} = a x_k$, $Q = 1$ $C = 1$

Then $P_{0|k}^{-1} = P_{0|k-1}^{-1} + a^{2k}$

Further $P_{0|0}^{-1} \approx 0$, so

6

$$P_{0|k}^{-1} = a^2 + a^4 + \cdots a^{2k} = a^2\left(\frac{1-a^{2k}}{1-a^2}\right)$$

If $a^2 < 1$ this converges to $\frac{a^2}{1-a^2}$.

If $a = .99$ $a^2 \approx 0.98$ & $P_{0|k}^{-1} = \frac{.98}{.02} = 49.$

So the smallest standard deviation of $x_0 = \frac{1}{7}$.

c) If $a = 1.01$ $a^2 = 1.02$

Then $P_{0|k}^{-1} = \frac{a^2(a^{2k}-1)}{a^2-1} = 51(e^{2k \log a} - 1)$

6

$$= 51(e^{(.04)k} - 1)$$

So the conditional standard deviation at k

$$= \left(\frac{1}{51(e^{.04k}-1)}\right)^{1/2} \longrightarrow 0 \text{ as } k \to \infty.$$

Solution

a)

$$E[\hat{A}_N \mid A, B] = \frac{1}{2N+1} \sum_{k=-N}^{N} E[A + Bk + W_k \mid A, B]$$

By the independence of the W_k and A & B, $E[W_k \mid A, B] = 0$.

So $$E[\hat{A}_N \mid A, B] = \frac{1}{2N+1} \sum_{k=-N}^{N} (A + Bk) = A.$$

Similarly
$$E[\hat{B}_N \mid A, B] = \frac{\sum_{k=1}^{N} k\,(A + Bk - A + Bk)}{2\sum_{k=1}^{N} k^2} = B.$$

b) Being conditionally unbiased the conditional mean of $\hat{A}_N$ is just A; so its variance is

7

$$E[(\hat{A}_N - A)^2 / A, B] = \frac{1}{(2N+1)^2} E[(\sum_{k=-N}^{N} W_k)^2 / A, B]$$

$$= \frac{1}{(2N+1)^2} (2N+1) Q = \frac{Q}{2N+1}$$

That of $\hat{B}_N$ is

$$E[(\hat{B}_N - B)^2 / A, B] = \frac{1}{4(\sum_1^N k^2)^2} E[(\sum_{k=1}^{N} (kW_k - kW_{-k}))^2 / A, B]$$

$$= \frac{2Q(\sum_1^N k^2)}{4(\sum_1^N k^2)^2} = \frac{Q}{2(\sum_1^N k^2)}$$

Finally $\hat{A}_N$ & $\hat{B}_N$ are uncorrelated since

$$E[(\hat{A}_N - A)(\hat{B}_N - B) / A, B] = \frac{E[(\sum_{k=-N}^{N} W_k)(\sum_{k=1}^{N}(kW_k - kW_{-k}))]}{(2N+1)\, 2 \sum_1^N k^2}$$

$$= \frac{Q \sum_{k=-N}^{N} k}{2(2N+1)\sum_1^N k^2} = 0.$$

Solution

3 c) $$E[(\bar{B}_N - B)^2 | A, B] = \frac{E[(W_N - W_{-N})^2]}{4N^2}$$

$$= \frac{Q}{2N^2}$$

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So the ratio

$$R_N = \frac{E[(\hat{B}_N - B)^2 | A, B]}{E[(\bar{B}_N - B)^2 | A, B]}$$

$$= \frac{2N^2}{Q} \cdot \frac{Q}{2 \sum_1^N k^2} = \frac{N^2}{\sum_1^N k^2}$$

But $\sum_1^N k^2 > \frac{N^3}{3}$. So

$$R_N < \frac{3N^2}{N^3} = \frac{3}{N}$$

which converges to zero as $N \to \infty$.

4 Solution

a) By the 'nesting' property of conditional expectations,

10 $$E[(x'_k - z_k)^2] = E[E[(x'_k - \hat{z}_{k|k} + \hat{z}_{k|k} - z_k)^2 \mid y_k, \dots]]$$

$$= E[(x'_k - \hat{z}_{k|k})^2 + E[(\hat{z}_{k|k} - z_k)^2 / y_k, \dots]]$$

$$= E[(x'_k - \hat{z}_{k|k})^2] + E[(\hat{z}_{k|k} - z_k)^2]$$

The second term is unaffected by the controls; so the optimal control law is the same as that for the given cost.

b) The Kalman filter for $z_{k|k}$ is , where $a = .99$

10 $$z_{k+1|k+1} = a z_{k|k} + \frac{P_{k+1|k}}{P_{k+1|k} + 1}(y_{k+1} - a\hat{z}_{k|k})$$

If this is time-invariant, $P_{k+1|k} = P$, where

$$P = a^2 P + 1 - \frac{a^2 P}{P+1} \quad \text{or} \quad ((a^2-1)P+1)(1+P) - a^2 P = 0$$

or $P^2 - a^2 P - 1 = 0$. As $P > 0$, $P = \frac{a^2}{2} + \frac{1}{2}\sqrt{a^4 + 4}$

If $a = .99$, $P = \frac{.98}{2} + \frac{1}{2}\sqrt{.96 + 4} = .49 + \sqrt{1.24}$
≈ 1.6

The Kalman gain $\frac{P}{P+1} = \frac{1.6}{2.6} = \frac{8}{13} = 0.6$

The noise term $y_{k+1} - a\hat{z}_{k|k} = w_{k+1} + z_{k+1} - \hat{z}_{k+1|k}$,
with variance $1 + P$

The term $\frac{P}{P+1}(y_{k+1} - a\hat{z}_{k|k})$ therefore has variance $\frac{P^2}{1+P}$

$$= \frac{(1.6)^2}{2.6} = \frac{8}{13} \times 1.6 = \frac{12.8}{13} \approx 1$$

Solution

a) The EKF is based on the approximate model

$$y_{k+1} = f(\hat{x}_{k|k}) + f'(\hat{x}_{k|k})(x_k - \hat{x}_{k|k}) + v_k$$

4 Then $\hat{x}_{k+1|k} = E(x_{k+1} / y_k, y_{k-1}, \dots) = f(\hat{x}_{k|k})$

b) For the statistical linearization filter

$$\hat{x}_{k+1|k} = E_{k|k}[f(x_k)]$$

$$P_{k+1|k} = E_{k|k}[(x_{k+1} - \hat{x}_{k+1|k})^2]$$

6 $$= E_{k|k}[R_k P^{-1}_{k|k} P_{k|k} P^{-1} R_k] + Q_s$$

$$= R_k P^{-1}_{k|k} R_k + Q_s.$$

c) If $f(x) = x - x^3$

$$\hat{x}_{k+1|k} = E_{k|k}[x_k - x_k^3] = \hat{x}_{k|k} - E_{k|k}[(\hat{x}_{k|k} + (x_k - \hat{x}_{k|k}))^3]$$

$$= \hat{x}_{k|k} - \hat{x}^3_{k|k} - 3P_{k|k}\hat{x}_{k|k}.$$

5 $$= (1 - 3P_{k|k})\hat{x}_{k|k} - \hat{x}^3_{k|k}$$

(Note that under the normality assumption, $E_{k|k}[(x_k - x_{k|k})^3] = 0$)

d) The prediction equation for the EKF is

$$\hat{x}_{k+1|k} = \hat{x}_{k|k} - \hat{x}^3_{k|k}.$$

5 If the conditional variance is small (e.g. if Q_0 is small), both filters would behaves similarly. However if $P_{k|k}$ is large then the statistical linearization filter is likely to be more accurate. The EKF is likely to produce estimates $\hat{x}_{k|k}$ that are biased away from the origin.

6 (Solution)

(a) Sufficient conditions for a unique $S \geq 0$ solving the control ARE are that: R is positive definite, the pair (A, B) is stabilizable and $(Q^{1/2}, A)$ is detectable.

3 (A more restrictive set of sufficient conditions is that $R > 0$, (A, B) is completely controllable and $(Q^{1/2}, A)$ completely observable)

(b) As $\Phi_N = S$, it follows that from the given Riccati difference equation and the ARE that $S_{N-1} = S$ and that, similarly,

$$S_k = S \quad \text{for all } k = 0, \ldots, N.$$

But by the quadratic cost identity, (with $\operatorname{cov}(x_0) = 0$, $x_0 = x$)

6
$$\min_u J^u_{0,N} = x^T S_0 x + \sum_{k=0}^{N-1} (0)^T (B^T S B + R)(0) + N \operatorname{tr}(S M M^T).$$

(take $u_k = -F_k x_k$)

$$= x^T S x + N \operatorname{trace}(S M M^T)$$

(c)
$$\lim_{N \to \infty} \frac{J^u_{0,N}(x)}{N} = \lim_{N \to \infty} \frac{x^T S x}{N} + \operatorname{trace}(S M M^T)$$
$$= \operatorname{trace}(S M M^T)$$

5 It can be shown that this is indeed the minimum value of $\lim_N \left(\frac{J^u_{0,N}}{N}\right)$ for all "stabilizing" $\{u_k\}$ for which $E(x_j^T Q x_j + u_j^T R u_j)$ is bounded in j.

(d) The separation principle is that the design of the control law and that of the Kalman filter can

6 be treated separately; the 'control' parameters B, Q, R do not affect the filter; M, C, N do not affect the optimal control law.