Paper Number(s): **E4.26**
C2.2

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

ESTIMATION AND FAULT DETECTION (MSc only in 2002)

Wednesday, 1 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Clark,J.M.C.

Second Marker(s): Allwright,J.C.

Corrected Copy

Special instructions for invigilators: **None**

Information for candidates:

Some formulae relevant to the questions

The normal $N(m,\sigma^2)$ density: $p(y)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\ (-\frac{(y-m)^2}{2\sigma^2})$

System equations:

$$x_{k+1}=Ax_k+Bu_k+Mv_k$$
$$y_k=Cx_k+Nw_k$$

Here, v_k and w_k are standard white-noise sequences with identity covariance matrices.

The Kalman one-step-ahead predictor equations:

$$\hat{x}_{k+1|k}=A\hat{x}_{k|k-1}+K(k)(y_k-C\hat{x}_{k|k-1})$$
$$K(k)=AP_{k|k-1}(CP_{k|k-1}C^T+NN^T)^{-1}$$
$$P_{k+1|k}=AP_{k|k-1}A^T+MM^T-AP_{k|k-1}C^T(CP_{k|k-1}C^T+NN^T)^{-1}CP_{k|k-1}A^T$$

The "completion of squares" identity for mean quadratic costs:

$$E[\sum_{k=0}^{N-1}(x_k^TQx_k)+x_N^TQ_Nx_N]$$
$$=E[x_0^TS_0x_0+\sum_{k=0}^{N-1}(u_k+F_kx_k)^T(B^TS_{k+1}B+R)(u_k+F_kx_k)]+\sum_{k=0}^{N-1}tr(S_{k+1}MM^T)$$

where for $k=0,\cdots,N-1$,

$$F_k=(B^TS_{k+1}B+R)^{-1}B^TS_{k+1}A$$
$$S_k=A^TS_{k+1}A+Q-A^TS_{k+1}B(B^TS_{k+1}B+R)^{-1}B^TS_{k+1}A,\quad S_N=Q_N$$

The algebraic Riccati equations:

$$S=A^TSA+Q-A^TSB(B^TSB+R)^{-1}B^TSA \qquad \text{(control)}$$

$$P=A^TPA+MM^T-APC^T(CPA^T+NN^T)^{-1}CPA^T \quad \text{(filtering)}$$

1. Suppose $x_1(t)$ is the indefinite integral of a coloured noise process $x_2(t)$ given by

$$\dot{x}_2 = -3x_2 + v\,,$$

where $v(t)$ is continuous-time Gaussian white noise for which

$$E[v(t)v(s)] = \delta(t-s)$$

(a) Show that the vector process $x(t) = (x_1(t), x_2(t))^T$ satisfies, for $t > s$, the integral equation

$$x(t) = \begin{bmatrix} 1 & -\tfrac{1}{3}e^{-3(t-s)} \\ 0 & -e^{-3(t-s)} \end{bmatrix} x(s) + \int_s^t \begin{bmatrix} 1 & -\tfrac{1}{3}e^{-3(t-r)} \\ 0 & -e^{-3(t-r)} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(r)dr$$

(Hint: first obtain the integral equation for $x_2(t)$ alone). [8 marks]

(b) Let x_k be the sampled process $x(kh)$. Show that x_k satisfies a difference equation of the form

$$x_{k+1} = \bar{A}x_k + \bar{v}_k$$

where $\bar{v}_k$ is discrete-time vector white noise. Determine $\bar{A}$ and the noise covariance $Q = E[v_k v_k^T]$.

Briefly explain why Q is non-singular. [7 marks]

(c) Suppose that a Kalman filter has been constructed that generates the current conditional means $\hat{x}_{k|k}$ of the sampled process from noisy measurements of its past and present values. Suppose that the value of $x(t)$ at the intermediate time $t = (k + \tfrac{1}{2})h$ is also of interest. Give an expression for its best estimate predicted at time kh. [5 marks]

2(a) Consider a sequence of observed random variables $Y_1, Y_2, \cdots, Y_n$ that are related to an unknown random variable X of mean m and variance p by

$$Y_k = X + N_k, \qquad k = 1, \cdots, n .$$

Here, each N_k is a zero-mean variable of variance q_k and the N_k and X are uncorrelated with each other. A feature of the linear least-squares estimate (LLSE) of a variable given a number of observed variables is that the error of estimation is uncorrelated with each of the observed variables. Use this characterization to establish that

$$\hat{X} = m + \frac{1}{p^{-1} + \sum_1^N q_k^{-1}} \left(\sum_1^N \left(\frac{Y_k - m}{q_k} \right) \right)$$

is the LLSE of X given $Y_1, \cdots, Y_n$ and determine the corresponding mean squared error $\hat{p}$.

[12 marks]

(b) An observer makes an initial estimate of the altitude of an aircraft as it passes directly overhead and then periodically estimates the altitude from measurements of its elevation as it recedes into the distance. The standard deviation of the initial estimate is 10 metres and that of the subsequent estimate computed from the k-th periodic measurement of the elevation is 5k metres. If it is assumed that the aircraft remains at a constant altitude, what is the best accuracy, expressed as a standard deviation, that the observer can hope to achieve in his estimation of the aircraft altitude?

[8 marks]

(Note: $\Sigma_1^{\infty} \frac{1}{k^2}$ is approximately 1.64) .

3. Suppose x_k and y_k are vector Gaussian processes satisfying the state space model

$$x_{k+1} = Ax_k + Bu_k + Mv_k \qquad x_0 \text{ normal}$$
$$y_k = Cx_k + Nw_k \ .$$

Here v_k and w_k are independent Gaussian white-noise sequences with zero means and identity covariances $E[v_k v_k^T]$, $E[w_k w_k^T]$. u_k depends on $y_k, y_{k-1}, \cdots$.

(a) Establish the expressions for the one-step-ahead predictor $\hat{x}_{k+1|k}$ and the Kalman gain $K(k)$ that are given on page one. (You may use the fact that for vector normal random variables

$$E[X \mid Y] = EX + Cov(X,Y)Cov(Y)^{-1}(Y - EY).)$$ **[8 marks]**

(b) Suppose z_k is a scalar controlled process described by

$$z_{k+1} = 0.2z_k + 0.3y_k + v_k$$

where y_k is an output measurement of the form

$$y_k = z_k + b + w_k \ .$$

Here, b is an unknown normal bias in the measurement and v_k and w_k are independent noise processes of unit variance.

An estimate of interest is the one-step-ahead predictor of z_k. Specify a state-space model from which an appropriate Kalman filter could be constructed.

[5 marks]

(c) Suppose, for the model in (b), the value of the bias b is known. Determine the steady-state form of a first-order Kalman filter that generates $\hat{z}_{k+1|k}$. **[7 marks]**

4. Consider a stochastic linear system

$$x_{k+1} = Ax_k + Bu_k + Mv_k \ , \qquad E[x_0] = 0, \ \ \mathrm{cov}[x_0] = P_0$$

and an averaged cost function of the special form

$$J_N^u = E[\sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) + x_N^T S x_N]$$

where v_k is standard white noise and where S is assumed to be a positive definite solution of the control algebraic Riccati equation (ARE) on page one.

(a) Using the "completion of squares" identity on page one, show that the control law that minimizes J_N^u over all control laws that are functions of the current state takes the time-invariant form

$$u_k = -Fx_k$$

where

$$F = (B^T SB + R)^{-1} B^T SA \, .$$

[8 marks]

(b) The "rate" cost $\bar{J}^u = \lim_{N\to\infty} \frac{1}{N} J_N^u$ is also minimized (over "stabilizing" control laws) by the control law in (a). Determine a formula for its optimal value.

[4 marks]

(c) Consider the scalar case where

$$x_{k+1} = x_k + u_k + mv_k \quad , \qquad x_0 = 0$$

and the "rate" cost is

$$\bar{J}^u = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E[2x_k^2 + 4u_k^2] \, .$$

Determine the optimal value of $\bar{J}^u$ and the steady-state variances of x_k and u_k generated by the optimal law.

[8 marks]

5. The behaviour of a scalar controlled process x_k is described by the equation

$$x_{k+1} = 0.5x_k + u_k + v_k, \qquad x_0 = 0 \tag{1}$$

where v_k is independent Gaussian white noise of variance σ^2.

(a) x_k is measured by a quantizer with an output y_k that takes values on a discrete set of levels $\{kh, k \text{ an integer}\}$. If the quantization parameter h is small compared with σ, the conditional probability density of x_k given y_k, $y_{k-1}, \cdots$, can reasonably be approximated by the corresponding conditional probability density of the variable

$$\alpha\hat{x}_{k|k-1} + (1-\alpha)y_k + b_k$$

where $\hat{x}_{k|k-1}$ is the predicted mean of x_k, $\alpha = \frac{h^2}{12\sigma^2}$ and b_k is a uniform random variable on the interval $[-\frac{h}{2}, \frac{h}{2}]$ independent of y_k. Show that the resulting conditional mean and variance of this variable are approximately the same as those of x_k given y_k, $y_{k-1}, \cdots$, where now

$$y_k = x_k + w_k \tag{2}$$

and w_k is Gaussian white noise, independent of x_k, with zero mean and variance $\frac{h^2}{12}$. It may be assumed that the predicted covariance of x_k is approximately σ^2. [10 marks]

(b) Suppose a cost $E[\Sigma_0^{N-1}(q(x_k)^2 + r(u_k)^2)]$ is to be minimised. Explain what is meant by the "separation principle". Demonstrate that it is valid under the assumption that observations $y_k, y_{k-1}, \cdots$ of the form (2) are available for the purposes of control at time k. If only the conditional means and covariances of x_k are available and this process may be conditionally non-Gaussian, as is the case in (a), describe how the separation principle has to be modified.

[10 marks]

6. The presence or absence of a fault in a piece of equipment is assessed by a number of measurements $y = (y_1, \cdots, y_N)^T$ made at different locations. If no fault is present ("$F = 0$"), the joint probability density of y is $p_0(y)$; if a fault is present ("$F = 1$"), the joint density is $p_1(y)$.

(a) If the prior probabilities of a fault are $P(F = 0) = \pi_0$, $P(F = 1) = \pi_1$, show that the Bayes test that minimizes the probability of error is:

choose $F = 1$ if $\frac{\pi_1 p_1(y)}{\pi_0 p_0(y)} \geq 1$;

choose $F = 0$ otherwise. [10 marks]

(b) Suppose that, if $F = f$ (= 0 or 1),

y_1 is normal with mean $2f$ and variance 1
y_2 is normal with mean 0 and variance $1 + f$
y_3 is uniformly distributed on $[2f, 2f + 4]$

that, conditional on F, the measurements y_1, y_2, y_3 are independent of each other and that $\pi_0 = \pi_1 = 1/2$.

Determine the outcome of the Bayes test in the two cases:

(i) $y_1 = 1$, $y_2 = 0$, $y_3 = 3$

(ii) $y_1 = 1$, $y_2 = 0$, $y_3 = 1$

and compute the probability of the decision being correct in the second case. [10 marks]

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1. Solution

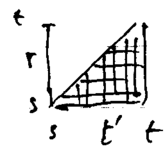
a) By the variation-of-constants formula

$$x_2(t) = e^{-3(t-s)} x_2(s) + \int_s^t e^{-3(t-r)} v(r)\, dr$$

But

$$x_1(t) = x_1(s) + \int_s^t x_2(t')\, dt'$$

$$= x_1(s) + \int_s^t e^{-3(t'-s)} x_2(s)\, dt'$$



8 $$+ \int_s^t \int_s^{t'} e^{-3(t'-r)} v(r)\, dr\, dt'$$

$$= x_1(s) + \frac{1}{3} e^{-3(t-s)} x_2(s)$$

$$+ \int_s^t \left(\int_r^t e^{-3(t'-r)} dt' \right) v(r)\, dr$$

$$= x_1(s) - \frac{1}{3} e^{-3(t-s)} x_2(s) + \int_s^t \left(-\frac{1}{3} e^{-3(t-r)} \right) v(r)\, dr$$

Combining the two gives the equation in (a).

b) Take $s = kh$ $t = (k+1)h$. From (a) we have

that $x_{k+1} = \bar{A} x_k + \bar{v}_k$ where

$$\bar{A} = \begin{bmatrix} 1 & -\frac{1}{3} e^{-3h} \\ 0 & -e^{-3h} \end{bmatrix}$$

7 $$Q = \int_s^t \begin{pmatrix} 1 & -\frac{1}{3} e^{-3(t-r)} \\ 0 & -e^{-3(t-r)} \end{pmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} e^{-3(t-r)} & -e^{-3(t-r)} \end{bmatrix} dr$$

$$= \int_s^t \begin{bmatrix} 0 & -\frac{1}{3} e^{-3(t-r)} \\ 0 & -e^{-3(t-r)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} e^{-3(t-r)} & -e^{-3(t-r)} \end{bmatrix} dr$$

$$= \int_0^h \begin{bmatrix} \frac{1}{9} e^{-6r} & \frac{1}{3} e^{-6r} \\ \frac{1}{3} e^{-6r} & e^{-6r} \end{bmatrix} dr = \frac{1}{6}(1-e^{-6h}) \begin{bmatrix} \frac{1}{9} & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$$

c) $$\hat{x}_{k+\frac{1}{2}|k} = E[\, x((k+\tfrac{1}{2})h) \mid \text{past observations} \,] = \begin{bmatrix} 1 & -\frac{1}{3} e^{-3h/2} \\ 0 & -e^{-3h/2} \end{bmatrix} \hat{x}_{k|k} .$$

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2 Solution

a) The estimation error is

$$X - \hat{X} = X - m - \frac{\Sigma_1^n (Y_k - m)/q_k}{p^{-1} + \Sigma_1^n q_k^{-1}}$$

The correlation with Y_k; since X, N_k are uncorrelated is:

8 $$E[(X-\hat{X})(Y_k - m)] = E\left[\left(X - m - \frac{\Sigma_1^n q_k^{-1}(X-m) + \Sigma N_k/q_k}{p^{-1} + \Sigma_1^n q_k^{-1}}\right)(X - m + N_k)\right]$$

$$= p - \frac{\Sigma_1^n q_k^{-1} p}{p^{-1} + \Sigma_1^n q_k^{-1}} - \frac{q_k/q_k}{p^{-1} + \Sigma_1^n q_k^{-1}}$$

$$= 0$$

This holds for all k; so $\hat{X}$ is the LLSE.

The mean squared error $E[(X-\hat{X})^2]$

$$= E\left[\left(\frac{p^{-1}(X-m) + \Sigma_1^n N_k/q_k}{p^{-1} + \Sigma_1^n q_k^{-1}}\right)^2\right]$$

4 $$= \frac{p^{-2} E(X-m)^2 + \Sigma_1^n E N_k^2/q_k^2}{(p^{-1} + \Sigma_1^n q_k^{-1})^2} = (p^{-1} + \Sigma_1^n q_k^{-1})^{-1}$$

b) Take the initial variance of altitude X to be $p = 100$

Variance of the measurement noise N_k is $q_k = 25 k^2$.

After n subsequent measurements, the LLSE $\hat{X}$ of X

8 has mean squared error (from part (a)) of

$$\frac{1}{\frac{1}{100} + \frac{1}{25}\Sigma_1^n \frac{1}{k^2}} = \frac{100}{1 + 4\Sigma_1^n 1/k^2}$$

As n increases, this decreases to $\frac{100}{1 + 4\Sigma_1^\infty 1/k^2} = \frac{100}{7.56}$

So the smallest standard deviation is $\frac{10}{\sqrt{7.56}} \approx 3.6$ metres

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3. (solution)

a) <u>Prediction</u> We may suppose x_k is conditionally Gaussian with mean $\hat{x}_{k/k}$ and covariance $P_{k/k}$. Then it follows that

$$\hat{x}_{k+1/k} = A\hat{x}_{k/k} + Bu_k$$
$$P_{k+1/k} = A P_{k/k} A^T + MM^T.$$

as u_k is a function of $(y_k, y_{k-1}, \cdots)$.

<u>Updating</u> Suppose we are conditioning on $y_{k-1}, \cdots$,

Then
$$\mathrm{Cov}(y_k / y_{k-1}, \cdots) = C P_{k/k-1} C^T + NN^T$$
$$\mathrm{Cov}(x_k, y_k / y_{k-1} \cdots) = P_{k/k-1} C^T$$

so

8 $$\hat{x}_{k/k} = \hat{x}_{k/k-1} + P_{k/k-1} C^T (C P_{k/k-1} C^T + NN^T)^{-1} \times (y_k - C\hat{x}_{k/k-1})$$

Combining this with the prediction equation gives

$$\hat{x}_{k+1/k} = A\hat{x}_{k/k-1} + Bu_k + \underbrace{A P_{k/k-1} C^T (C P_{k/k-1} C^T + NN^T)^{-1}}_{= K(k).} (y_k - C\hat{x}_{k/k-1})$$

b) Take $x_k^1 = z_k$, $x_k^2 = b$. The state-space model that is suitable for filtering is then.

5 $$x_{k+1} = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{pmatrix} 0.3 \\ 0 \end{pmatrix} y_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_k$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + w_k.$$

c) If b is known we can replace y_k by $\bar{y}_k = y_k - b = z_k + w_k$.
$p = P_{k/k-1}$ (for z_k) satisfies the ARE. where $a = 0.2$.

$$p = a^2 p + 1 - \frac{a^2 p}{p+1}$$

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or $0.96 p(p+1) = p + 1 - 0.04p \; : 0.96 p^2 = 1$

$\Rightarrow \; p = 1.02 \quad \Rightarrow \; K = \dfrac{(1.02)\,0.2}{2.02} = 0.101$ and

$$\hat{x}_{k+1/k} = .2\,\hat{x}_{k/k-1} + 0.101(y_k - b - \hat{x}_{k/k-1}).$$

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4. Solution

(a) By the completion-of-squares identity

$$J_N^u = tr(SP_0) + \sum_{k=0}^{N-1} tr(S_{k+1} MM^T) + E\left[\sum_{k=0}^{N-1} (u_k + F_k x_k)^T (B^T S_{k+1} B + R)(u_k + F_k x_k) \right]$$

8 Since u_k may be a function of x_k, $u_k = -F_k x_k$ annihilates the last term and minimizes J_N^u.

The ARE implies that the solution to the difference Riccati eqn. is $S_0 = S_1 = \dots = S_N = S$ and therefore that

$$F_k = (B^T S B + R)^{-1} B^T S A.$$

(b) The optimal value of J_N^u in (a) is

$$J_N^0 = tr(SP_0) + N\, tr(SMM^T)$$

4 The optimal rate cost is then

$$\bar{J}^0 = \lim_{N \to \infty} \frac{1}{N} J_N^0 = \underline{tr(SMM^T)}$$

(c) The ARE becomes

$$S = S + 2 - \frac{S^2}{S+4}$$

or

$$S^2 - 2S - 8 = 0$$

8 As $S > 0$ $\quad S = 1 + \sqrt{9} = 4$

and so $\quad \underline{\bar{J}^0 = 4m^2}$

The optimal control $u_k = -Fx_k = -\frac{S}{S+4} x_k = -\frac{1}{2} x_k$

Therefore $x_{k+1} = \frac{1}{2} x_k + m v_k$

$Cov(x_{k+1}) = Cov(x_k)$ and $Cov(x_{k+1}) = \frac{1}{4} Cov(x_k) + m^2$

gives

$$Cov(x_k) = \underline{\frac{4}{3} m^2}$$

$$Cov(u_k) = Cov\left(-\frac{1}{2} x_k\right) = \underline{\frac{m^2}{3}}$$

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5. solution

(a) The variable

$$\alpha \hat{x}_{k|k-1} + (1-\alpha) y_k + b_k$$

clearly has the conditional mean $\alpha \hat{x}_{k|k-1} + (1-\alpha) y_k$ and conditional variance $\frac{1}{h}\int_{-h/2}^{h/2} x^2\,dx = \frac{h^2}{12}$

If y_k is given by (2) the 'updating' formula of filtering gives

10

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \frac{\mathrm{Cov}(x_k, y_k \mid y_{k-1}, \dots)}{\mathrm{Cov}(y_k \mid y_{k-1}, \dots)}(y_k - \hat{x}_{k|k-1})$$

$$= \hat{x}_{k|k-1} + \frac{P_{k|k-1}}{P_{k|k-1} + h^2/12}(y_k - \hat{x}_{k|k-1})$$

with $P_{k|k-1} \approx \sigma^2$,

$$= \frac{\alpha \hat{x}_{k|k-1}}{1+\alpha} + \frac{y_k}{1+\alpha}$$

$$\approx \alpha \hat{x}_{k|k-1} + (1-\alpha) y_k$$

Similarly $\mathrm{Cov}(x_k \mid y_k, y_{k-1}, \dots) = \dfrac{h^2/12}{1+\alpha} \approx \dfrac{h^2}{12}$

b) The separation principle states that in the linear-quadratic-Gaussian set-up, the optimal control law is linear in the best estimate and the coefficient F_k depends only on the 'control' parameters such as Q, R. (and A and B). The best estimate + filter design depends only on statistical parameters such as σ^2, h^2 and not on control parameters.

10 The key expression to be minimized in the completion-of-squares formula is

$$\sum_{k=0}^{N-1} E\left[(u_k + F_k x_k)^T (B^T S_{k+1} B + R)(u_k + F_k x_k)\right]$$

$$= \sum_{k=0}^{N-1} E\left[E\left[(\cdots) \mid y_k, y_{k-1}, \dots\right]\right]$$

$$= \sum_k E\left[(u_k + F_k \hat{x}_{k|k})^T (B^T S_{k+1} B + R)(u_k + F_k \hat{x}_{k|k})\right] + \mathrm{tr}\left(P_{k|k}(B^T S_{k+1} B + R) F_k F_k^T\right)$$

which is minimized if $u_k = -F_k \hat{x}_{k|k}$.

In the non-Gaussian case, if the minimization is over linear (or affine) function of $y_k, y_{k-1}, \dots$, the principle still holds.

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6 Solution

(a) Let $\delta(y)$ denote the Bayes decision rule, with values 0 or 1.

Then the probability of error

$$= P[\,\delta(y) \neq F\,]$$

$$= P(\,\delta(y) = 1 \mid F=0)\,\pi_0 + P(\,\delta(y)=0 \mid F=1)\,\pi_1$$

$$= \pi_0 \int_D p_0(y)\,dy + \pi_1 \int_{D^c} p_1(y)\,dy \quad \text{where } D = \{y : \delta(y)=1\}$$

10 $$= \pi_0 \int_D p_0(y)\,dy + \pi_1 \Big(1 - \int_D p_1(y)\,dy\Big)$$

$$= \pi_0 \int_D \Big(1 - \frac{\pi_1 p_1(y)}{\pi_0 p_0(y)}\Big)\, p_0(y)\,dy + \pi_1$$

This is minimized if $D = \{y : 1 - \frac{\pi_1 p_1(y)}{\pi_0 p_0(y)} \le 0\}$, which gives the result.

(b) $$\frac{p_1(y_1)}{p_0(y_1)} = \exp\Big(-\frac{(y_1-2)^2}{2} + \frac{y_1^2}{2}\Big) = \exp(2y_1 - 2).$$

$$\frac{p_1(y_2)}{p_0(y_2)} = \sqrt{\tfrac{1}{2}}\,\exp\Big(-\frac{y_2^2}{4} + \frac{y_2^2}{2}\Big) = \sqrt{\tfrac{1}{2}}\,\exp\frac{y_2^2}{4}$$

10 $$\frac{p_1(y_3)}{p_0(y_3)} = 0 \quad \text{if } 0 \le y_3 \le 2$$

$$= 1 \quad \text{if } 2 < y_3 \le 4$$

$$= \infty \quad \text{if } 4 < y_3 < 6$$

As the y_1, y_2, y_3 are conditionally independent, $\frac{p_1(y)}{p_0(y)}$ is $\prod_{k=1}^{3} \frac{p_1(y_k)}{p_0(y_k)}$.

(i) $$\frac{p_1(y)}{p_0(y)} = 1 \cdot \sqrt{\tfrac{1}{2}} \cdot 1$$. So $\delta(y)=0$; $F=0$ is chosen

(ii) $$\frac{p_1(y)}{p_0(y)} = 0 \quad \text{since} \quad \frac{p_1(y_3)}{p_0(y_3)} = 0$$

So $F=0$ is chosen

Clearly $P(\,\delta(y) = F = 0 \mid y_3 = 1) = 1.$