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C4.1
ISE4.23

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

MSc and EEE/ISE PART IV: M.Eng. and ACGI

DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS

Friday, 2 May 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible   First Marker(s) : I.M. Jaimoukha
Second Marker(s) : J.M.C. Clark

Special Information for Invigilators : None

Information for Candidates : None

1. Consider the descriptor realization

$$\begin{aligned}\hat{E}\dot{x}(t) &= \hat{A}x(t) + \hat{B}u(t)\\ \hat{F}y(t) &= \hat{C}x(t) + \hat{D}u(t)\end{aligned}$$

where

$$\hat{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 4 \end{bmatrix},$$

$$\hat{F} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 8 & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) Derive a state-space realization

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

and determine the corresponding transfer matrix $G(s)$. **[4]**

(b) Find the uncontrollable and/or unobservable modes of the realization in (a) and determine whether the realization is detectable and stabilizable. **[4]**

(c) Find a minimal realization for $G(s)$. **[4]**

(d) Find the McMillan form of $G(s)$ and determine the pole and zero polynomials. What is the McMillan degree of $G(s)$? **[4]**

(e) Determine the system zeros, indicating the type of each zero. **[4]**

2. (a) Define internal stability for the feedback loop in Figure 2.1, and derive necessary and sufficient conditions for which this loop is internally stable. [4]

(b) Suppose that $G(s)$ is stable. Give a parameterization of all internally stabilizing controllers for $G(s)$. [4]

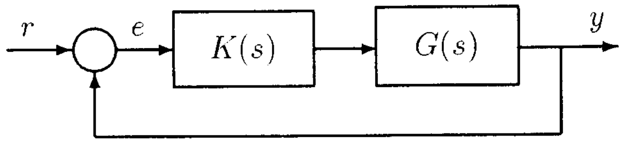

Figure 2.1

(c) In the Internal Model Control design procedure illustrated in Figure 2.2 below, $G(s)$ represents a plant, $G_o(s)$ is a nominal model of the plant and $P(s)$ is a compensator. Here

$$G_o(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

i. Suppose that there is no uncertainty in the plant description so that $G(s) = G_o(s)$. Using the answer to part (b), derive necessary and sufficient conditions on $P(s)$ so that the loop in Figure 2.2 is internally stable. [4]

ii. Suppose now that there is an output multiplicative uncertainty in the description of the plant so that $G(s) = [I + \Delta(s)]G_o(s)$ with $\Delta(s)$ a stable transfer matrix satisfying

$$\|\Delta(j\omega)\| \leq |1 + j\omega|^2, \ \forall \omega \in \mathcal{R}.$$

Let $S(s)$ denote the transfer matrix from r to $r + y$ in Figure 2.2. Design a controller $P(s)$ which internally stabilizes the feedback loop in Figure 2.2 for all $\Delta(s)$, and such that $\|S(0)\| \leq 0.1$. [8]

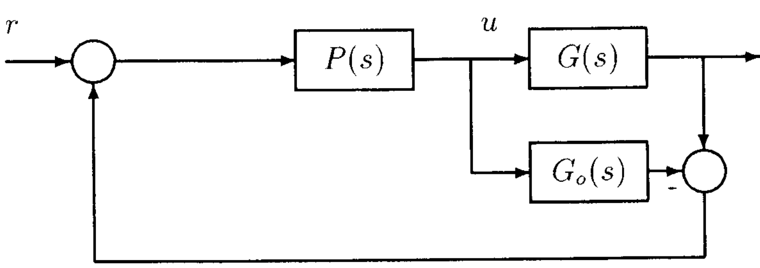

Figure 2.2

3. Figure 3.1 illustrates the implementation of the Kalman filter,

$$\dot{x}_e = Ax_e + Bu + K(y - Cx_e),$$

for the linear dynamics

$$\dot{x} = Ax + B(u + w), \qquad y = Cx + v.$$

Here, w and v are uncorrelated white noises with covariances $W = I$ and $V = I$, respectively, and $K = PC^T$, where P is the stabilizing solution to

$$AP + PA^T - PC^TCP + BB^T = 0.$$

Assume that the triple (A, B, C) is minimal. Define $G(s) = C(sI - A)^{-1}$.

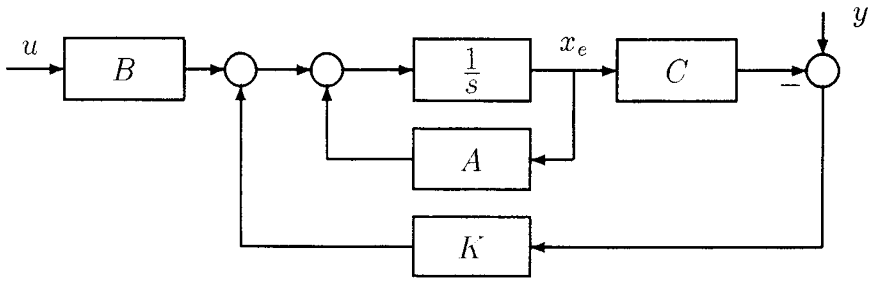

Figure 3.1

(a) Let $L(s) = I + G(s)K$. Show that

$$L(s)L(-s)^T = I + G(s)BB^TG(-s)^T.$$ **[5]**

(b) Derive the smallest upper bounds on $\|(I + GK)^{-1}\|_\infty$ and $\|(I + GK)^{-1}GK\|_\infty$ guaranteed by Part (a). **[5]**

(c) Suppose that stable perturbations Δ_1 and Δ_2 are introduced as shown in Figure 3.2. Here, $G(s)$ and K are as above. Using the answer to Part (b), derive the maximal stability radius (using the $\mathcal{L}_\infty$-norm as a measure):

(i) for $\Delta_1(s)$ when $\Delta_2(s) = 0$, **[5]**

(ii) for $\Delta_2(s)$ when $\Delta_1(s) = 0$. **[5]**

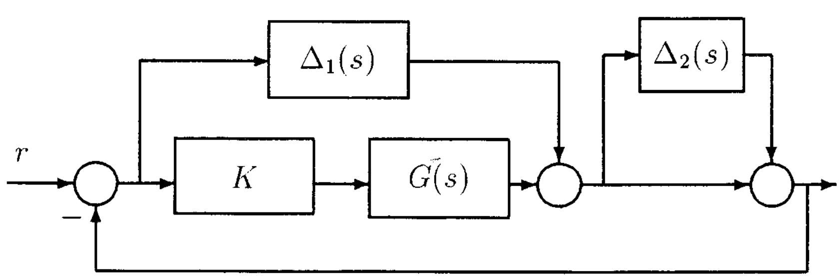

Figure 3.2

4. (a) State the small gain theorem concerning the internal stability of a feedback loop having a forward transfer matrix Δ and a feedback transfer matrix S. **[4]**

(b) Consider the feedback loop shown in Figure 4 where $G(s)$ represents a plant model and $K(s)$ represents an internally stabilizing compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|cc} -10/3 & 2/5 & 0 & 1 & -1 \\ 2/5 & -5/2 & -7/6 & 3/5 & 4/5 \\ 0 & -7/6 & -1 & 1 & 1 \\ \hline 1 & 3/5 & 1 & 0 & 0 \\ -1 & 4/5 & 1 & 0 & 0 \end{array}\right] \in \mathcal{RH}_\infty.$$

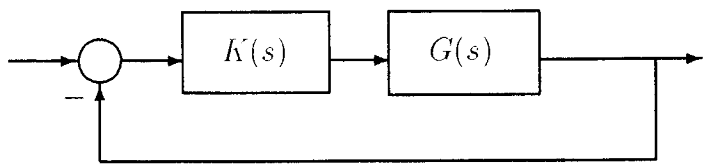

Figure 4

(i) Show that the given realization for $G(s)$ is balanced and evaluate the Hankel singular values of $G(s)$. **[6]**

(ii) Design a first order internally stabilizing controller $K(s)$ for $G(s)$ as follows:

- Replace $G(s)$ in Figure 4 by a first order approximation $G_1(s)$ and give an upper bound on $\|G(s) - G_1(s)\|_\infty$. What is the transfer matrix for $G_1(s)$?

- Find the set of all internally stabilizing controllers for the new feedback loop.

- Using the small gain theorem and the bound on $\|G(s) - G_1(s)\|_\infty$, choose a first order internally stabilizing controller for the feedback loop of Figure 4. **[10]**

5. Consider the feedback configuration in Figure 5.1. Here, $G(s)$ is a nominal plant model and $K(s)$ is a compensator. The signals $r(s)$ and $n(s)$ represent the reference and sensor noise, respectively. The design specifications are to synthesize a compensator $K(s)$ such that the feedback loop is internally stable and:

- For good tracking, it is required that, when $n(s) = 0$,

$$\|e(j\omega)\| < |w_1(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega.$$

- To limit the control effort, it is required that when $n(s) = 0$,

$$\|u(j\omega)\| < |w_2(j\omega)^{-1}|\|r(j\omega)\|, \forall\omega.$$

- For good sensor noise attenuation it is required that, when $r(s) = 0$,

$$\|y(j\omega)\| < |w_3(j\omega)^{-1}|\|n(j\omega)\|, \forall\omega,$$

where $w_1(s)$, $w_2(s)$ and $w_3(s)$ are suitable filters.

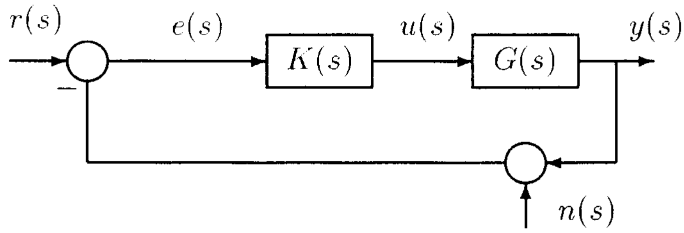

Figure 5.1

(a) Derive $\mathcal{H}_\infty$-norm bounds, in terms of $G(s), K(s), w_1(s)$, $w_2(s)$ and $w_3(s)$ that are sufficient to achieve the design specifications. **[6]**

(b) Derive a generalized regulator formulation of the design problem that captures the sufficient conditions in Part (a). **[7]**

(c) Assume that $K(s)$ achieves the design specifications in Part (a). Suppose that an uncertainty $\Delta(s)$ is introduced as in Figure 5.2 where $\Delta(s)$ is a stable transfer matrix. Derive an upper bound on $\|\Delta(j\omega)\|, \forall\omega$, for which robust stability is guaranteed. **[7]**

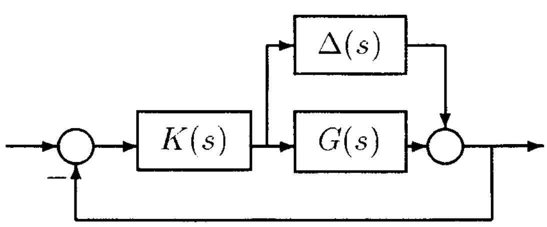

Figure 5.2

6. Consider the regulator shown in Figure 6 for which it is assumed that the triple (A, B, C) is minimal and $x(0)=0$.

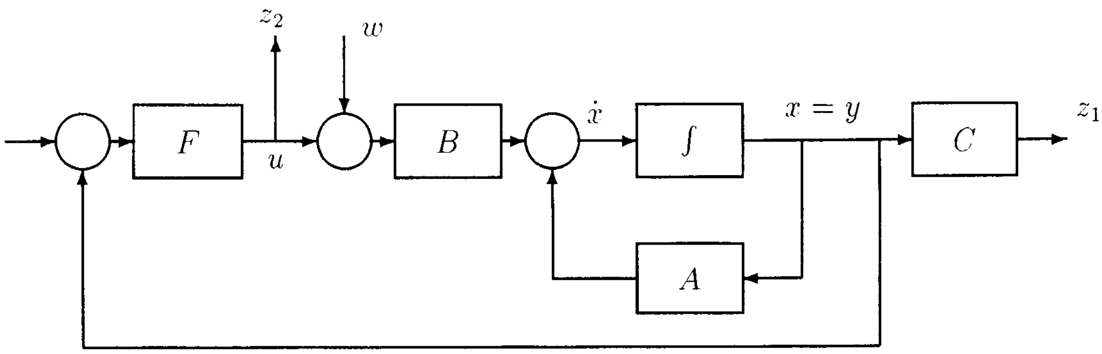

Figure 6

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and let H denote the transfer matrix from w to z. A stabilizing state-feedback gain matrix F is to be designed such that, for given $\gamma > 0$, $\|H\|_\infty < \gamma$.

(a) Write down the generalized regulator system for this design problem. **[6]**

(b) By using the Lyapunov function $V(t) = x(t)^T X x(t)$, where X is to be determined, derive sufficient conditions for the solution of the design problem. Your conditions should be in the form of the existence of a certain solution to an algebraic Riccati equation. It should also include an expression for F and an expression for the worst-case disturbance w.
Use the identity

$$(\alpha R - \alpha^{-1} S)^T (\alpha R - \alpha^{-1} S) = \alpha^2 R^T R + \alpha^{-2} S^T S - R^T S - S^T R,$$

for scalar $\alpha \neq 0$ and matrices R and S to complete the squares. **[10]**

(c) Suggest an algorithm for evaluating the optimal value for γ guaranteed by the sufficient conditions of Part (b). **[4]**

[END]