Master – 25/4/03

Design of Linear Multivariable Control Systems

Solutions 2002/2003

1. (a) Multiplying the first and second descriptor equations from the left by $\hat{E}^{-1}$ and $\hat{F}^{-1}$, respectively we get the state-space realization with

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 4 \\ \hline 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 1 \end{array}\right].$$

(b) Since $[A - sI \quad B]$ loses rank for $s = -3$, $\boxed{-3 \text{ is an uncontrollable mode,}}$ and since $[A^T - sI \quad C^T]$ loses rank for $s = 4$, $\boxed{4 \text{ is an unobservable mode.}}$ Since the uncontrollable mode is stable, $\boxed{\text{the realization is stabilizable}}$ and since the unobservable mode is unstable, $\boxed{\text{the realization is not detectable.}}$

(c) By removing the uncontrollable and unobservable modes we get the minimal realization

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{bmatrix} = \frac{1}{s-1}\begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}.$$

(d) By performing the following elementary operations: (1) $r_1 \leftrightarrow r_2$, (2) $r_2 := r_2 - (s+1)r_1$, (3) $c_2 := c_2 - (s+1)c_1$, (4) $c_2 = -c_2$, the McMillan form of $G(s)$ is given by,

$$G(s) = \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 & s+1 \\ 0 & -1 \end{bmatrix} =: L(s)M(s)R(s),$$

where $L(s)$ and $R(s)$ are unimodular.
The pole and zero polynomials are given by

$$\boxed{p(s) = s-1,} \quad \& \quad \boxed{z(s) = s+3}$$

respectively. $\boxed{\text{The McMillan degree is 1}}$ since it is equal to the degree of the pole polynomial.

(e) Since $s = -3$ is an uncontrollable mode, $\boxed{-3 \text{ is an input decoupling zero.}}$ Since $s = 4$ is an unobservable mode, $\boxed{4 \text{ is an output decoupling zero.}}$ It follows from Part (d) that the system has a $\boxed{\text{transmission zero at } s = -3.}$

2. (a) Inject a signal d in between $G(s)$ and $K(s)$ and call the input to $G(s)$ u. The loop is internally stable if and only if the transfer matrix from $\begin{bmatrix} d \\ r \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if $T^{-1}(s)$ is stable.

(b) Since $G(s)$ is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

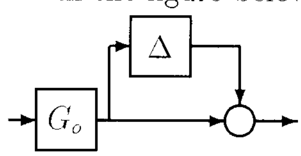
Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I-GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I-GK)^{-1} \\ 0 & (I-GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since $(I - GK)^{-1} = I + GK(I - GK)^{-1}$, it follows that if G is stable, then the loop is internally stable if and only if $Q := K(I - GK)^{-1}$ is stable. Rearranging terms shows that K is internally stabilizing if and only if $K = Q(I + GQ)^{-1}$ for some stable Q.

(c) i. Setting $G = G_o$, the transfer matrix between $(r + y)$ and u in Figure 2.2 is given by $(I + PG)^{-1}P$. Comparing this with Figure 2.1 and the answer to Part (b), it follows that we can identify K with $(I + PG)^{-1}P$ and P with Q. It follows that the loop is internally stable if and only if P is stable.

ii. Set $G = (I + \Delta)G_o$ as shown in the figure below.



Since K is internally stabilizing, $K = P(I + GP)^{-1}$ for some stable P from Part (b). We search for a stable P to satisfy the design requirements. Let the input to Δ be ϵ while the output from Δ be δ. Then a simple calculation shows that $\epsilon = C\delta$ where $C = (I - GK)^{-1}GK$ is the complementary sensitivity which is stable. Now $S = (I - GK)^{-1} = I + GP$ and $C = GK(I - GK)^{-1} = GP$. The small gain theorem implies that for K to stabilize the loop in Figure 2.2 for all Δ, we must have $\|G(j\omega)P(j\omega)\| < \frac{1}{|1+j\omega|^2}$, so we choose

$$P(s) = h\frac{1}{(s+1)^2}\,G^{-1}(s) = h \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

where $-1 < h < 1$ is to be determined. Since $S(0) = I + G(0)P(0) = (1 + h)I_2$, it follows that any $-1 < h \leq -0.9$ will satisfy the design specifications.

3. (a) By direct expansion, $\boxed{\text{using } K = PC^T}$,

$$L(s)L(-s)^T = I + C(sI-A)^{-1}PC^T + CP(-sI-A^T)^{-1}C^T + C(sI-A)^{-1}PC^TCP(-sI-A^T)^{-1}C^T.$$

$\boxed{\text{Using the Riccati equation}}$,

$$PC^TCP = AP + PA^T + BB^T = -(sI-A)P - P(-sI-A^T) + BB^T.$$

$\boxed{\text{Multiplying by } C(sI-A)^{-1} \text{ from the left and } (-sI-A^T)^{-1}C^T \text{ from the right}}$,

$$C(sI-A)^{-1}PC^T + CP(-sI-A^T)^{-1}C^T + C(sI-A)^{-1}PC^TCP(-sI-A^T)^{-1}C^T = C(sI-A)^{-1}BB^T(-sI-A^T)^{-1}C^T,$$

and the result follows.

(b) Part (a) implies that $\underline{\sigma}[I + G(j\omega)K] \geq 1,\ \forall\, \omega \in \mathcal{R}$. It follows that

$$\boxed{\|(I+GK)^{-1}\|_\infty \leq 1.}$$

Now, $(I+GK)^{-1}GK = L(L^{-1} - I) = I - L^{-1}$. Thus, Part (a) implies that

$$\bar{\sigma}[L(j\omega)^{-1} - I] \leq 1 + \bar{\sigma}[L(j\omega)^{-1}] \leq 1 + \frac{1}{\underline{\sigma}[L(j\omega)]} \leq 2,$$

so that

$$\boxed{\|(I+GK)^{-1}GK\|_\infty \leq 2.}$$

(c) (i) Set $\Delta_2 = 0$. Let ϵ be the input to Δ_1 and δ be the output of Δ_1. Then

$$\epsilon = -(\delta + GK\epsilon) = -(I+GK)^{-1}\delta$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_1(I+GK)^{-1}\|_\infty < 1$. But Part (b) implies that $\|(I+GK)^{-1}\|_\infty \leq 1$. This shows that the loop will tolerate perturbations of size

$$\boxed{\|\Delta_1\|_\infty < 1}$$

without losing internal stability.

(ii) Set $\Delta_1 = 0$. Let ϵ be the input to Δ_2 and δ be the output of Δ_2. Then

$$\epsilon = -GK(\delta + \epsilon) = -(I+GK)^{-1}GK\delta.$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if $\|\Delta_2(I+GK)^{-1}GK\|_\infty < 1$. But Part (b) implies that $\|(I+GK)^{-1}GK\|_\infty < 2$. This shows that the loop will tolerate perturbations Δ_2 of size

$$\boxed{\|\Delta_2\|_\infty < 0.5}$$

without losing internal stability.

4. (a) Suppose that both $\boxed{\Delta(s) \text{ and } S(s) \text{ are stable}}$. Then the feedback loop with forward transfer matrix $\Delta(s)$ and feedback transfer matrix $S(s)$ is stable if

$$\boxed{\|\Delta(s)S(s)\|_\infty < 1.}$$

(b) (i) The realization is balanced if

$$\boxed{A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0}$$

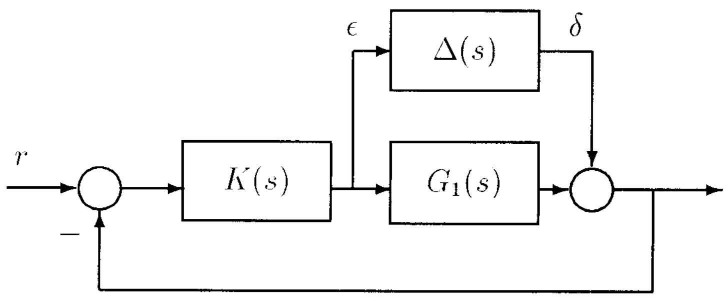
for $\boxed{\Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3) > 0}$ and where the $\sigma_i's$ are the Hankel singular values of $K(s)$. A simple calculation gives

$$\Sigma = \text{diag}(0.3, 0.2, 1) \quad \Rightarrow \quad \boxed{[\sigma_1, \sigma_2, \sigma_3] = [1, .3, .2].}$$

(ii) Let $G_1(s)$ denote a first order balanced truncation of $G(s)$. Then $G_1(s) = G(s) + \Delta(s)$ where

$$\boxed{G_1(s) = \tfrac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \|\Delta\|_\infty \leq 2\sum_{i=2}^{3} \sigma_i = 1}$$

Then replacing $G(s)$ by $G_1(s)$ in the loop of Figure 4 is equivalent to:



Now

$$\epsilon = -K(I + G_1K)^{-1}\delta$$

and so the loop is stable if $\|K(I + G_1K)^{-1}\|_\infty \|\Delta\|_\infty < 1$. from the small gain theorem. Since $\|\Delta\|_\infty \leq 1$ $\boxed{\text{it is sufficient that } \|K(I + G_1K)^{-1}\|_\infty < 1}$. However, since $G_1(s)$ is stable, the set of all internally stabilizing controllers for $G_1(s)$ is given by:

$$\boxed{K = Q(I - G_1Q)^{-1}}$$

for stable Q. Furthermore,

$$K(I + G_1K)^{-1} = Q.$$

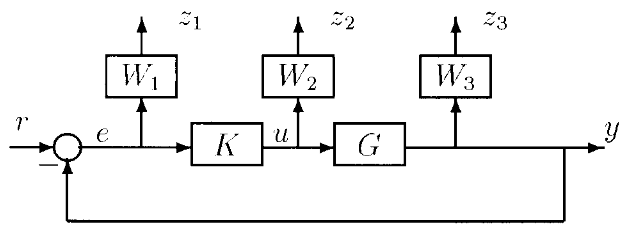
Thus we can take $\boxed{Q = qI_2}$ where $\boxed{q \text{ is constant}}$ (to guarantee a first order controller) and $\boxed{|q| < 1}$ (to guarantee stabilization of G).

5. (a) It is clear that we require K to be internally stabilizing.

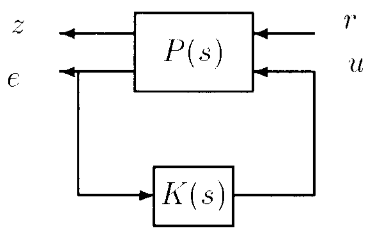
- A simple calculation shows that, when $n(s) = 0$, $e(s) = -S(s)r(s)$ where $S(s) = [I+G(s)K(s)]^{-1}$ is the sensitivity. Thus $\|e(j\omega)\| \le \|S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the first design specification is $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall\omega$ or equivalently $\boxed{\|W_1 S\|_\infty < 1}$, where $W_1 = w_1 I$.
- A similar calculation shows that, when $n(s) = 0$, $u(s) = -K(s)S(s)r(s)$. Thus $\|u(j\omega)\| \le \|K(j\omega)S(j\omega)\|\|r(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|K(j\omega)S(j\omega)\| < |w_2^{-1}(j\omega)|$, $\forall\omega$ or equivalently $\boxed{\|W_2 KS\|_\infty < 1}$, where $W_2 = w_2 I$.
- When $r(s) = 0$, a similar calculation shows that $y(s) = -C(s)n(s)$ where $C(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$ is the complementary sensitivity. Thus $\|y(j\omega)\| \le \|C(j\omega)\|\|n(j\omega)\|$. It follows that a sufficient condition to achieve the second design specification is $\|C(j\omega)\| < |w_3^{-1}(j\omega)|, \forall\omega$ or equivalently $\boxed{\|W_3 C\|_\infty < 1}$, where $W_3 = w_3 I$.

To satisfy all design requirements, it is sufficient that $\boxed{\left\| \begin{bmatrix} W_1 S \\ W_2 KS \\ W_3 C \end{bmatrix} \right\|_\infty < 1}$.

(b) The design specifications reduce to the requirement that the transfer matrix from r to $z = [z_1^T \ z_2^T \ z_3^T]^T$ in the following diagram has $\mathcal{H}_\infty$-norm less than 1.



The corresponding generalized regulator formulation is to find an internally stabilizing K such that $\|\mathcal{F}_l(P, K)\|_\infty < 1$:



$$\boxed{P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} W_1 & -W_1 G \\ 0 & W_2 \\ 0 & W_3 G \\ \hline I & -G \end{array}\right].}$$

(c) Let the input to Δ be ϵ and the output from Δ be δ. Then $\epsilon = -KS\delta$ and since KS is stable, the small gain theorem implies closed-loop stability if $\|\Delta(j\omega)K(j\omega)S(j\omega)\| < 1, \forall\omega$. Since K achieves the design specifications of Part (a), $\boxed{\|\Delta(j\omega)\| < |w_2(j\omega)|, \forall\omega}$ is the maximal stability radius.

6. (a) The generalized regulator formulation is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s)\begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \; u(s) = Fy(s), \; P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c||c|c} A & B & B \\ \hline C & 0 & 0 \\ 0 & 0 & I \\ \hline I & 0 & 0 \end{array}\right].$$

(b) The requirement $\|H\|_\infty < \gamma$ is equivalent to $J := \|z\|_2^2 - \gamma^2\|w\|_2^2 < 0$, with $\|v\|_2^2 := \int_0^\infty \|v(t)\|^2 dt$. Let $V = x^T X x$ and set $u = Fx$. Provided that $X = X^T > 0$ and $\dot V < 0$ along closed loop trajectory, we can assume $\lim_{t\to\infty} x(t) = 0$. Then

$\dot V = \dot x^T X x + x^T X \dot x = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x.$

Integrating from 0 to ∞ and using $x(0) = x(\infty) = 0$,

$$0 = \int_0^\infty [x^T\left(A^T X + XA + F^T B^T X + XBF\right)x + x^T XBw + w^T B^T X x]dt.$$

Using the definition of J and adding the last equation, $J =$

$$\int_0^\infty \{x^T[A^T X + XA + C^T C + F^T F + F^T B^T X + XBF]x - [\gamma^2 w^T w - x^T XBw - w^T B^T X x]\}dt.$$

Completing the squares by using

$$(F + B^T X)^T (F + B^T X) = F^T F + F^T B^T X + XBF + XBB^T X$$

$$\|(\gamma w - \gamma^{-1} B^T X x)\|^2 = \gamma^2 w^T w - w^T B^T X x - x^T XBw + \gamma^{-2} x^T XBB^T X x,$$

$$J = \int_0^\infty \{x^T[A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X]x + \left\|(F + B^T X)x\right\|^2 - \left\|\gamma w - \gamma^{-1} B^T X x\right\|^2\}dt.$$

Thus two sufficient conditions for $J < 0$ are the existence of X such that

$$\boxed{A^T X + XA + C^T C - (1-\gamma^{-2})XBB^T X = 0,} \qquad \boxed{X = X^T > 0.}$$

The state feedback gain is $F = -B^T X$ and the worst case disturbance is $w^* = \gamma^{-2} B^T X x$. The closed-loop with these feedback laws is $\dot x = [A - (1-\gamma^{-2})BB^T X]x$ and a third condition is therefore $\boxed{Re\ \lambda_i[A - (1-\gamma^{-2})BB^T X] < 0,\ \forall\ i.}$

It remains to prove $\dot V < 0$ along state-trajectory with $u = Fx$ and $w = 0$. But

$$\boxed{\dot V = x^T\left(A^T X + XA + F^T B^T X + XBF\right)x = -x^T(C^T C + (1+\gamma^{-2})XBB^T X)x < 0}$$

for all $x \neq 0$ (since (A, B, C) is assumed minimal) proving closed-loop stability.

(c) The optimal γ is the smallest value of $\gamma > 0$ such that the sufficient conditions are satisfied. This can be calculated by a **binary search algorithm** as follows:

i. Choose upper and lower bound γ_u and γ_l

ii. Define $\gamma = 0.5(\gamma_u + \gamma_l)$

iii. If there exists a positive stabilizing solution to the Riccati equation set $\gamma_u = \gamma$ else set $\gamma_l = \gamma$.

iv. Go to ii.