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IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE  
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2001

MSc and EEE/ISE PART IV: M.Eng. and ACGI

**DESIGN OF LINEAR MULTIVARIABLE CONTROL SYSTEMS**

Thursday, 3 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners: Jaimoukha,I.M. and Weiss,G.

**Corrected Copy**

**Special instructions for invigilators:**

None

**Information for candidates:**

None

1. Let the transfer matrix  $G(s)$  have a state space realisation

$$G(s) \stackrel{s}{=} \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left[ \begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ \hline 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array} \right]$$

- (a) Find the uncontrollable and/or unobservable modes and determine whether the realisation is detectable and stabilisable. [4]
- (b) Suppose that  $K \in \mathcal{R}^{2 \times 4}$  and  $L \in \mathcal{R}^{4 \times 2}$  are arbitrary matrices. Determine two of the eigenvalues of  $A - BK$  and two of the eigenvalues of  $A - LC$ . Explain how you arrive at your answer. [4]
- (c) Find a minimal realisation for  $G(s)$ . [4]
- (d) Find the McMillan form of  $G(s)$  and determine the pole and zero polynomials. What is the McMillan degree of  $G(s)$ ? [9]
- (e) Determine the system zeros, indicating the type of each zero. [4]

2. (a) Define internal stability for the feedback loop in Figure 1, and derive necessary and sufficient conditions for which this loop is internally stable. [6]

- (b) Suppose that  $G(s)$  is given by

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

Give a parameterisation of all internally stabilising controllers for  $G(s)$ . [7]

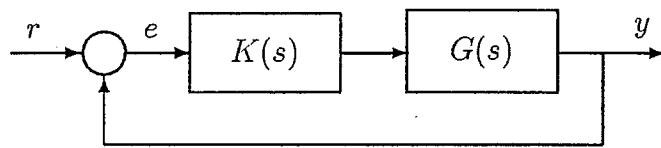


Figure 1

- (c) Let  $G(s)$  be as given in Part (b) and let  $S(s)$  denote the transfer matrix from  $r$  to  $e$  in Figure 1. Suppose now that an uncertainty on  $G(s)$  is introduced as shown in Figure 2, with  $\Delta(s)$  a stable transfer matrix satisfying

$$\|\Delta(j\omega)\| \leq |1 + j\omega|^2, \quad \forall \omega \in \mathcal{R},$$

Using the answer to Part (b) design a controller  $K(s)$  which internally stabilises the feedback loop in Figure 2 for all  $\Delta(s)$ , and such that  $\|S(0)\| \leq 0.1$ . [12]

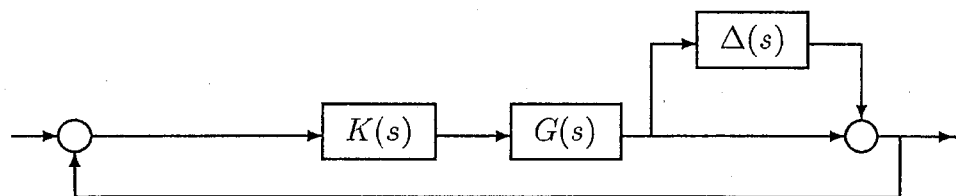


Figure 2

3. (a) State Gershgorin's Theorem concerning the location of the eigenvalues of a complex matrix. Let

$$A = \begin{bmatrix} -4 & 0 & 0 & \delta_1 \\ 0 & -3 & \delta_2 & 0 \\ 0 & \delta_2 & -2 & 0 \\ \delta_1 & 0 & 0 & -1 \end{bmatrix} \in \mathcal{R}^{4 \times 4}.$$

Using Gershgorin's Theorem, give a range of values of  $\delta_1$  and  $\delta_2$  for which  $A$  is guaranteed to have all its eigenvalues in the left half of the complex plane.

[7]

- (b) For the feedback loop in Figure 3, let  $K$  be a constant diagonal matrix. State a Nyquist type stability criterion in terms of the direct Nyquist array of  $G(s)$ .

[6]

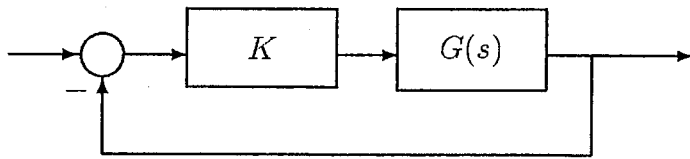


Figure 3

- (c) Consider the feedback loop in Figure 4. Here

$$G(s) = \begin{bmatrix} \frac{5}{s-1} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

and  $\Delta(s)$  is a transfer matrix representing a structured uncertainty of the form

$$\Delta(s) = \begin{bmatrix} 0 & \delta_{12}(s) \\ \delta_{21}(s) & 0 \end{bmatrix}$$

such that  $\|\Delta\|_\infty < 1$ . Use the answer to Part (b) to derive the range of values of  $k_1$  and  $k_2$  for which the closed-loop system is guaranteed to be internally stable.

[12]

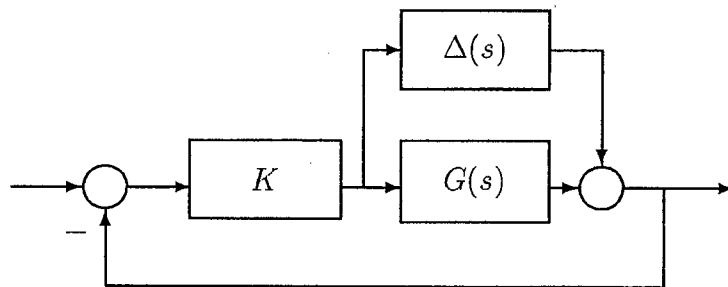


Figure 4

4. Figure 5 illustrates the implementation of the control law  $u(t) = -Kx(t) + r(t)$  which (when  $r(t) = 0$ ) minimises

$$J(x_0, u) = \int_0^{\infty} \|Cx(t)\|^2 + \|u(t)\|^2 dt$$

subject to  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $x(0) = x_0$  where  $K = B'P$  and  $P = P'$  is the unique stabilising solution of the Riccati equation  $A'P + PA - PBB'P + C'C = 0$ . Here,  $T'$  denotes the complex conjugate transpose of  $T$ . Assume that the triple  $(A, B, C)$  is minimal. Let  $F(s) = (sI - A)^{-1}B$ ,  $G(s) = C(sI - A)^{-1}B$  and  $L(s) = I + KF(s)$ .

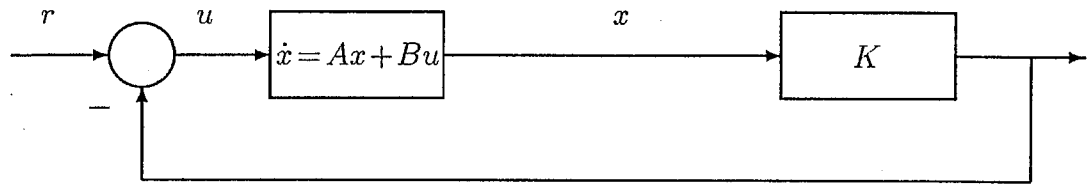


Figure 5

- (a) Show that

$$L(j\omega)'L(j\omega) = I + G(j\omega)'G(j\omega). \quad [8]$$

- (b) Suppose that  $G(s) = \frac{4}{s+3}$ . Derive a balanced, minimal state-space realisation  $G(s) = C(sI - A)^{-1}B$  and evaluate  $K$  for this realisation. [5]

- (c) Let  $G(s)$  and  $K$  be as in Part (b). Suppose a stable uncertainty  $\Delta$  is introduced as shown in Figure 6. Derive the maximal stability radius (using the  $\mathcal{H}_\infty$ -norm as a measure) for  $\Delta$  that can be deduced from Part (a) and the small gain theorem. [12]

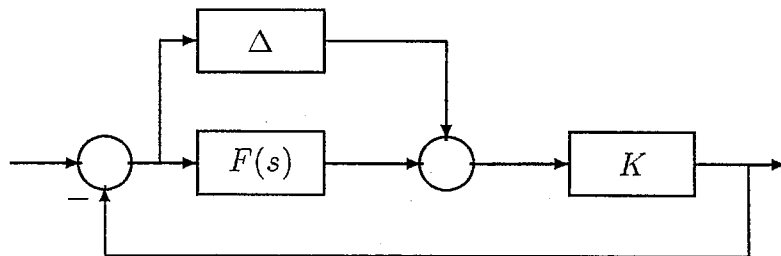


Figure 6

5. Consider the feedback loop shown in Figure 7 where  $G(s)$  represents a plant model and  $K(s)$  represents an internally stabilising compensator. Suppose that

$$G(s) \stackrel{s}{=} \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{cc|cc} -4 & -4 & 2 & 2 \\ -4 & -5 & 1.2 & 1.6 \\ \hline 2 & 1.2 & 0 & 0 \\ 2 & 1.6 & 0 & 0 \end{array} \right] \in \mathcal{RH}_\infty.$$

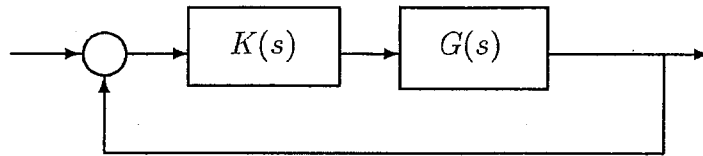


Figure 7

- (a) Show that the given realisation for  $G(s)$  is balanced and evaluate the Hankel singular values of  $G(s)$ .

[6]

- (b) Design a family of first order internally stabilising controllers  $K(s)$  for  $G(s)$  using the following procedure:

- i. Replace  $G(s)$  in Figure 7 by a first order approximation  $G_r(s)$  and give an upper bound on  $\|G(s) - G_r(s)\|_\infty$ .
- ii. Find the set of all internally stabilising controllers for the new feedback loop.
- iii. Using the small gain theorem and the bound on  $\|G(s) - G_r(s)\|_\infty$ , choose a family of first order internally stabilising controllers for the feedback loop of Figure 7.

[12]

- (c) Design a non-dynamic internally stabilising controller  $K$  for  $G(s)$  such that  $\|K\| \geq 1$ .

[7]

(Hint: Use the procedure outlined in Part (b) and the fact that  $G_r(s)$  is rank deficient.)

6. Consider the feedback configuration in Figure 8. Here,  $G(s)$  is a nominal plant model and  $K(s)$  is a compensator. The signals  $r(s)$  and  $n(s)$  are the Laplace transforms of the reference and sensor noise, respectively. The design specifications are to synthesise a compensator  $K(s)$  such that the feedback loop is internally stable and:

- For good tracking, it is required that, when  $n(s) = 0$ ,

$$\frac{\|e(j\omega)\|}{\|r(j\omega)\|} < |w_1(j\omega)^{-1}|, \forall \omega.$$

- For good sensor noise attenuation it is required that, when  $r(s) = 0$ ,

$$\frac{\|y(j\omega)\|}{\|n(j\omega)\|} < |w_2(j\omega)^{-1}|, \forall \omega$$

where  $w_1(s)$  is a low pass and  $w_2(s)$  is a high pass filter.

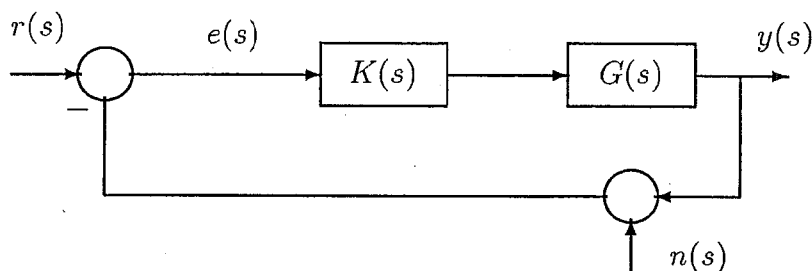


Figure 8

- Derive  $\mathcal{H}_\infty$ -norm bounds, in terms of  $G(s)$ ,  $K(s)$ ,  $w_1(s)$  and  $w_2(s)$  that are sufficient to achieve the design specifications. [8]
- Derive a generalised regulator formulation of the design problem that captures the sufficient conditions in Part (a). [8]
- Assume that a compensator  $K(s)$  achieves the design specifications in Part (a). Suppose now that an uncertainty  $\Delta(s)$  is introduced as shown in Figure 9. Assume that  $\Delta(s)$  is a stable transfer matrix. Derive the maximal stability radius for  $\|\Delta(j\omega)\|, \forall \omega$ . [9]

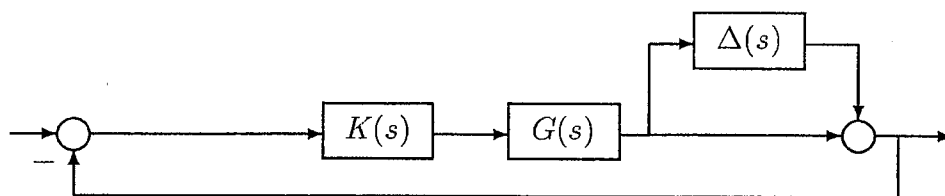


Figure 9



# SOLUTIONS - LINEAR MULTIVARIABLE CONTROL 2001

1. (a) Since  $[A - sI \ B]$  loses rank for  $s = 3$  and  $s = 5$ , they are uncontrollable modes, and since  $[A^T - sI \ C^T]$  loses rank for  $s = 4$  and  $s = 5$ , they are unobservable modes. Since the uncontrollable modes are unstable, the realisation is not stabilisable, and since the unobservable modes are unstable, the realisation is not detectable. [4]

(b) Since the modes  $\lambda = 3$  and  $\lambda = 5$  are uncontrollable, they cannot be assigned via state feedback and so they are eigenvalues of  $A - BK$ . Similarly, since  $\lambda = 4$  and  $\lambda = 5$  are unobservable modes, they cannot be assigned via output injection and so they are eigenvalues of  $A - LC$ . [4]

(c) By removing the uncontrollable and/or unobservable modes we get the minimal realisation

$$G(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} 1 & 1 & 2 \\ \hline 2 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] = \left[ \begin{array}{cc} \frac{s+1}{s-1} & \frac{4}{s-1} \\ \frac{1}{s-1} & \frac{s+1}{s-1} \end{array} \right] = \frac{1}{s-1} \begin{bmatrix} s+1 & 4 \\ 1 & s+1 \end{bmatrix}. \quad [4]$$

(d) By performing the following elementary operations: (1)  $r_1 \leftrightarrow r_2$ , (2)  $r_2 := r_2 - (s+1)r_1$ , (3)  $c_2 := c_2 - (s+1)c_1$ , (4)  $c_2 = -c_2$ , the McMillan form of  $G(s)$  is given by,

$$G(s) = \begin{bmatrix} s+1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 & s+1 \\ 0 & -1 \end{bmatrix} =: L(s)M(s)R(s),$$

where  $L(s)$  and  $R(s)$  are unimodular.

The pole and zero polynomials are given by

$$p(s) = s - 1, \quad z(s) = s + 3$$

respectively. The McMillan degree is the degree of the pole polynomial and is therefore equal to 1. [9]

(e) Since  $s = 3$  and  $s = 5$  are uncontrollable modes, they are input decoupling zeros. Since  $s = 4$  and  $s = 5$  are unobservable modes, they are output decoupling zeros. So,  $s = 5$  is an input/output decoupling zero. It follows from Part (d) that the system has a transmission zero at  $s = -3$ . [4]

2. (a) Inject a signal  $d$  in between  $G(s)$  and  $K(s)$  and call the input to  $G(s)$   $u$ . The loop is internally stable if and only if the transfer matrix from  $\begin{bmatrix} d \\ r \end{bmatrix}$  to  $\begin{bmatrix} u \\ e \end{bmatrix}$  is stable (no poles in the closed right half plane). Since

$$\begin{bmatrix} d \\ r \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} u \\ e \end{bmatrix} =: T(s) \begin{bmatrix} u \\ e \end{bmatrix}$$

the loop is internally stable if and only if  $T^{-1}(s)$  is stable. [6]

- (b) Since  $G(s)$  is stable, we proceed as follows. Note that

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}.$$

Hence

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -K \\ 0 & I - GK \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix}^{-1} = \begin{bmatrix} I & K(I - GK)^{-1} \\ 0 & (I - GK)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix}.$$

Finally, since  $(I - GK)^{-1} = I + GK(I - GK)^{-1}$ , it follows that if  $G$  is stable, then the loop is internally stable if and only if  $Q := K(I - GK)^{-1}$  is stable. Rearranging terms shows that  $K$  is internally stabilising if and only if  $K = Q(I + GQ)^{-1}$  for some stable  $Q$ . [7]

- (c) Since  $K$  is required to be internally stabilising,  $K = Q(I + GQ)^{-1}$  for some stable  $Q$  from Part (b). We search for a stable  $Q$  to satisfy the design requirements. Let the input to  $\Delta$  be  $\epsilon$  while the output from  $\Delta$  be  $\delta$ . Then a simple calculation shows that  $\epsilon = C\delta$  where  $C = (I - GK)^{-1}GK$  is the complementary sensitivity which is stable. Now

$$\begin{aligned} S &= (I - GK)^{-1} = I + GK(I - GK)^{-1} = I + GQ, \\ C &= GK(I - GK)^{-1} = GQ. \end{aligned}$$

The small gain theorem implies that for  $K$  to stabilise the loop in Figure 2 for all  $\Delta$ , we must have

$$\|G(j\omega)Q(j\omega)\| < \frac{1}{|1 + j\omega|^2}$$

so we choose

$$Q(s) = h \frac{1}{(s+1)^2} G^{-1}(s) = h \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

where  $-1 < h < 1$  is to be determined. Since

$$S(0) = I + G(0)Q(0) = (1 + h)I_2$$

it follows that any  $-1 < h \leq -0.9$  will satisfy the design specifications. [12]

3. (a) Gershgorin's Theorem: Let  $A$  be any  $n \times n$  complex matrix. The eigenvalues of  $A$  lie in  $\mathcal{D}_1$ , the union of the discs,

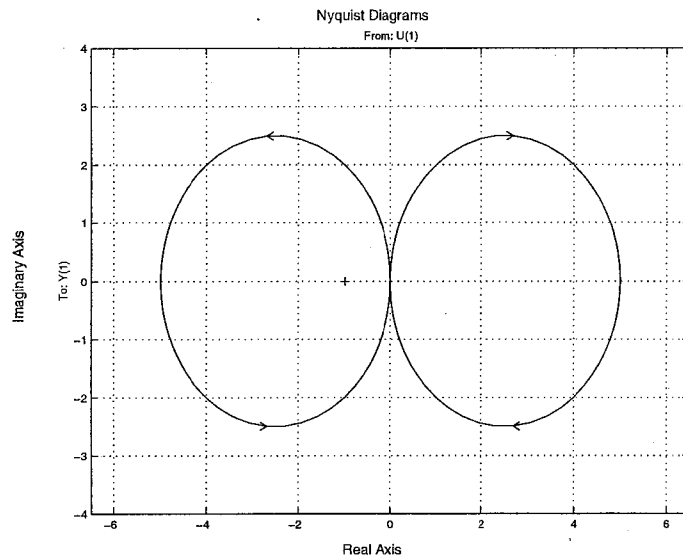
$$|l - m_{ii}| \leq \sum_{j \neq i} |m_{ij}|, \quad i = 1, \dots, n,$$

and they also lie in  $\mathcal{D}_2$ , the union of the discs,

$$|l - m_{ii}| \leq \sum_{j \neq i} |m_{ji}|, \quad i = 1, \dots, n.$$

For the given  $A$ , taking column sums, the Gershgorin discs are centred on  $-4, -3, -2$  and  $-1$  and have radii  $|\delta_1|, |\delta_2|, |\delta_2|$  and  $|\delta_1|$ , respectively. Thus  $\mathcal{D}_2$  lies in the closed left half plane if  $-1 < \delta_1 < 1$  and  $-2 < \delta_2 < 2$ . [7]

- (b) Let  $G(s)$  have  $P$  closed right half plane poles. Assume that  $K^{-1} + G(s)$  is diagonally dominant, that is,  $|\frac{1}{k_i} + G_{ii}(s)| \geq \sum_{j \neq i} |G_{ji}(s)|$ , for all  $i$  and for all  $s$  on the Nyquist contour. Let the  $i$ th Gershgorin band of  $G(s)$  encircle the point  $-\frac{1}{k_i}$  a total of  $N_i$  times anticlockwise. Then the loop is internally stable if and only if  $\sum_i N_i = P$ . [6]



- (c) For the given  $G(s)$ ,  $P = 1$ . The Nyquist plots for  $G_{11}$  (left circle) and  $G_{22}$  (right circle) are shown above. Since  $\|\Delta\|_\infty < 1$ , it follows that  $\|\delta_{12}\|_\infty < 1$  and  $\|\delta_{21}\|_\infty < 1$ . Thus the Gershgorin circles all have radius 1 at most. To guarantee stability, it is sufficient that the number of anticlockwise encirclements by the Gershgorin bands of  $G_{11}$  of  $-\frac{1}{k_1}$  and the number of anticlockwise encirclements by the Gershgorin bands of  $G_{22}$  of  $-\frac{1}{k_2}$  is 1. Thus the closed-loop system is stable if  $-4 < -\frac{1}{k_1} < -1$  (equivalently, if  $0.25 < k_1 < 1$ ) and if  $-\frac{1}{k_2} > 6$  (equivalently, if  $k_2 > -\frac{1}{6}$ ) or  $-\frac{1}{k_2} < -1$  (equivalently, if  $k_2 < 1$ ). [12]

4. (a) By direct evaluation,  $L(j\omega)'L(j\omega) =$

$$I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' + B'(-j\omega I - A')^{-1}K'K(j\omega I - A)^{-1}B.$$

But

$$K'K = A'P + PA + C'C = -(-j\omega I - A')P - P(j\omega I - A) + C'C$$

from the Riccati equation. So,  $L(j\omega)'L(j\omega)$

$$\begin{aligned} &= I + K(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}K' \\ &\quad + B'(-j\omega I - A')^{-1}[-(-j\omega I - A')P - P(j\omega I - A) + C'C](j\omega I - A)^{-1}B \\ &= I + [K - B'P](j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}[K' - PB] \\ &\quad + B'(-j\omega I - A')^{-1}C'C(j\omega I - A)^{-1}B \\ &= I + G(j\omega)'C'CG(j\omega). \end{aligned}$$

[8]

(b) A minimal, balanced state-space realisation of  $G(s)$  is given by

$$G(s) \stackrel{s}{=} \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{c|c} -3 & 2 \\ \hline 2 & 0 \end{array} \right].$$

The Riccati equation becomes

$$-3P - 3P - 4P^2 + 4 = 0$$

which has a stabilising solution  $P = 0.5$ . Hence  $K = B'P = 1$ .

[5]

(c) Let  $\epsilon$  be the input to  $\Delta$  and  $\delta$  be the output of  $\Delta$ . Then

$$\epsilon = -K(\delta + G\epsilon) = -(I + KG)^{-1}K\delta.$$

Using the small gain theorem (since the regulator is stable and the perturbation is assumed stable), the loop is stable if  $\|\Delta(I + KG)^{-1}K\|_\infty < 1$ . But Part (a) implies that  $\underline{\sigma}[I + KG(j\omega)] \geq 1$  which implies  $\|(I + KG)^{-1}\|_\infty \leq 1$ . Furthermore,  $K = 1$  from Part (b). Hence the loop will tolerate perturbations of size (measured in the  $\mathcal{H}_\infty$  norm) at least 1 without losing internal stability, since  $\|\Delta\|_\infty < 1$  implies that

$$\|\Delta(I + KG)^{-1}K\|_\infty < 1.$$

[12]

5. (a) The realisation of  $G(s)$  is balanced if

$$A\Sigma + \Sigma A' + BB' = A'\Sigma + \Sigma A + C'C = 0$$

for  $\Sigma = \text{diag}(\sigma_1, \sigma_2) > 0$  and where the  $\sigma_i$ 's are the Hankel singular values of  $G(s)$ . A simple calculation gives  $\Sigma = \text{diag}(1, 0.4)$ . [6]

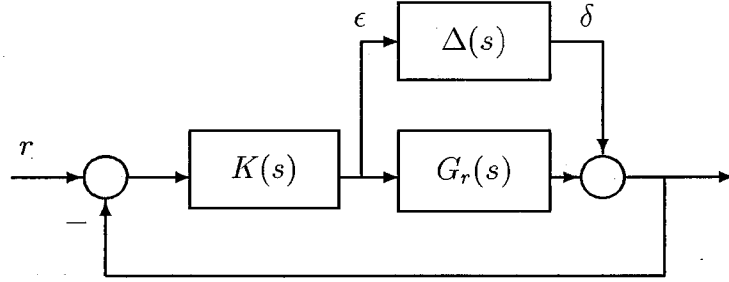
- (b) Let

$$G_r(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -4 & 2 & 2 \\ \hline 2 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right] = \frac{4}{s+4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

denote a first order balanced truncation of  $G(s)$ . Then  $G_r(s) = G(s) + \Delta(s)$  where

$$\|\Delta\|_\infty \leq 2 \sum_{i=2}^2 \sigma_i = 0.8.$$

Then replacing  $G(s)$  by  $G_r(s)$  in the loop of Figure 7 is equivalent to:



Now  $\epsilon = -K(I + G_r K)^{-1} \delta$  and so the small gain theorem implies that the loop is stable if  $\|\Delta K(I + G_r K)^{-1}\|_\infty < 1$  which is guaranteed if  $\|K(I + G_r K)^{-1}\|_\infty < 1.25$  since  $\|\Delta\|_\infty \leq 0.8$ . However, the set of all internally stabilising controllers for  $G_r(s)$  is given by  $K = Q(I + G_r Q)^{-1}$  for stable  $Q$ . Furthermore,  $K(I + G_r K)^{-1} = Q$ . Thus we can take  $Q = qI_2$  where  $q$  is constant (to guarantee a first order controller) and  $|q| < 1.25$  (to guarantee stabilisation of  $G$ ): For example, taking  $q = -1$  gives

$$K(s) \stackrel{s}{=} \left[ \begin{array}{c|cc} -3 & 1 & 1 \\ \hline 1 & -1 & 0 \\ 1 & 0 & -1 \end{array} \right]. \quad [12]$$

- (c) Arguing as in Part (b), the set of all internally stabilising controllers for  $G_r(s)$  is given by  $K = Q(I + G_r Q)^{-1}$  for stable  $Q$ . Since  $G_r$  has rank 1, we can ensure that  $K$  is non-dynamic by choosing non-dynamic  $Q$  such that  $G_r Q = 0$ . A possible choice is

$$Q = q \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

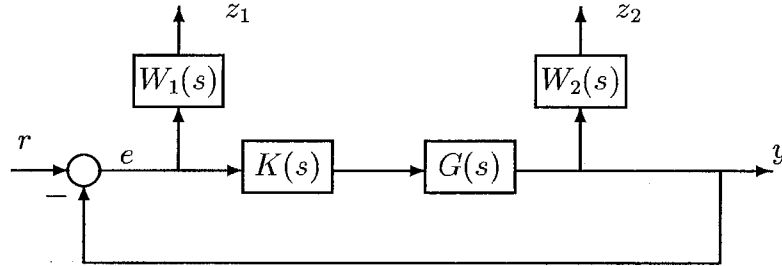
giving  $K = Q$ . Finally, to guarantee stabilisation of  $G$  and to ensure that  $\|K\| \geq 1$  we choose  $q$  such that  $1 \leq \|Q\| = \sqrt{2}|q| < 1.25$ , or  $\frac{1}{\sqrt{2}} \leq |q| < \frac{1.25}{\sqrt{2}}$ . [7]

6. (a) It is clear that we require  $K$  to be internally stabilising.

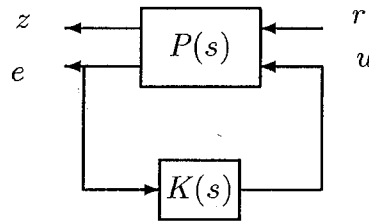
- A simple calculation shows that, when  $n(s) = 0$ ,  $e(s) = -S(s)r(s)$  where  $S(s) = [I + G(s)K(s)]^{-1}$  is the sensitivity. Thus  $\|e(j\omega)\| \leq \|S(j\omega)\| \|r(j\omega)\|$ . It follows that a sufficient condition to achieve the first design specification is  $\|S(j\omega)\| < |w_1^{-1}(j\omega)|, \forall \omega$  or equivalently  $\|W_1 S\|_\infty < 1$ , where  $W_1 = w_1 I$ .
- When  $r(s) = 0$ , a similar calculation shows that  $y(s) = -C(s)n(s)$  where  $C(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$  is the complementary sensitivity. Thus  $\|y(j\omega)\| \leq \|C(j\omega)\| \|n(j\omega)\|$ . It follows that a sufficient condition to achieve the second design specification is  $\|C(j\omega)\| < |w_2^{-1}(j\omega)|, \forall \omega$  or equivalently  $\|W_2 C\|_\infty < 1$ , where  $W_2 = w_2 I$ .

Thus, to satisfy both design requirements, it is sufficient (but not necessary) that  $\left\| \begin{bmatrix} W_1 S \\ W_2 C \end{bmatrix} \right\|_\infty < 1$ . [8]

(b) The design specifications reduce to the requirement that the transfer matrix from  $r$  to  $z = [z_1^T \ z_2^T]^T$  in the following diagram has  $\mathcal{H}_\infty$ -norm less than 1.



The corresponding generalised regulator formulation is to find an internally stabilising  $K$  such that  $\|\mathcal{F}_l(P, K)\|_\infty < 1$ :



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[ \begin{array}{c|c} W_1 & -W_1 G \\ \hline 0 & W_2 G \\ \hline I & -G \end{array} \right]. \quad [8]$$

(c) Let the input to  $\Delta$  be  $\epsilon$  while the output from  $\Delta$  be  $\delta$ . Then  $\epsilon = -C\delta$  where  $C = (I + GK)^{-1}GK$  is the complementary sensitivity which is stable. Using the small gain theorem, closed-loop stability is assured provided that  $\|\Delta(j\omega)C(j\omega)\| < 1, \forall \omega$ . Since  $K(s)$  achieves the design specifications of Part (a),  $\|\Delta(j\omega)\| < |w_2^{-1}(j\omega)|, \forall \omega$  gives the maximal stability radius. [9]