IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

MSc and EEE PART IV: MEng and ACGI

## LINEAR OPTIMAL CONTROL

Friday, 29 April 10:00 am

Time allowed: 3:00 hours

**There are SIX questions on this paper.**

**Answer FOUR questions.**

## Corrected Copy

*All questions carry equal marks*

**Any special instructions for invigilators and information for candidates are on page 1.**

| Examiners responsible | First Marker(s) : | A. Astolfi |
|---|---|---|
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Special instructions for invigilators:                          None

Information for candidates:

Hamilton Jacobi theory:    $\dot{x}(t) = f(t, x, u)$, $x(0) = x_0$

$$J(x_0, u) = \int_\tau^T L(t, x, u)dt + m(x(T)),$$

$$-\frac{\partial V}{\partial t} = \min_u \left[ L(t, x, u) + \frac{\partial V}{\partial x} f(t, x, u) \right], \ V(x, T) = m(x)$$

Linear Quadratic Regulator:    $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$

$$J(x_0, u) = \int_\tau^T \left[ x(t)'Qx(t) + u(t)'Ru(t) \right] dt + x(T)'Mx(T)$$

$$Q = Q' \geq 0, \ R = R' > 0, \ M = M' \geq 0$$

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P, \ P(T) = M$$

$$u(t) = -R^{-1}B'Px(t) = -Kx(t).$$

The matrices $A$, $B$, $Q$, $R$, $P$ and $K$ may depend upon $t$.

Minimum principle:    $\dot{x} = f(x, u)$, $u \in \mathcal{U}$

$$J(x_0, u) = \int_0^{t_f} L(x(t), u(t))dt,$$

$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda' f(x, u),$$

$$\dot{\lambda}^\star = -\frac{\partial H}{\partial x}\bigg|'_{(x^\star, u^\star, \lambda_0^\star, \lambda^\star)},$$

$$H(x^\star, \omega, \lambda_0^\star, \lambda^\star) \geq H(x^\star, u^\star, \lambda_0^\star, \lambda^\star), \ \forall \omega \in \mathcal{U},$$

$$H(x^\star, u^\star, \lambda_0^\star, \lambda^\star) = k \quad \text{for fixed } t_f$$
$$H(x^\star, u^\star, \lambda_0^\star, \lambda^\star) = 0 \quad \text{for free } t_f$$

1. Consider the system with one-dimensional state

$$\dot{x} = f(x) + u$$

with initial state $x_0 > 0$ and with the cost to be minimised

$$J(x_0, u) = \int_0^\infty L(x, u) dt$$

where

$$L(x, u) = (q(x))^2 + u^2.$$

(a) Write the Hamilton-Jacobi equation associated with this optimal control problem.
(Hint: use the fact that the value function $V(x, t)$ is independent of $t$). [2]

(b) Solve the Hamilton-Jacobi equation derived in part (a).
(Hint: note that the Hamilton-Jacobi equation is quadratic in $\frac{\partial V}{\partial x}$. Hence you can solve for $\frac{\partial V}{\partial x}$, and then integrate formally in $x$.) [8]

(c) Assume $f(x) = -q(x)$ and $q(x) = x + x^3$. Compute a positive definite solution of the Hamilton-Jacobi equation which is also such that $V(0) = 0$. [4]

(d) For $f(x)$ and $q(x)$ as in part (c), compute the optimal control and the optimal closed-loop system. [2]

(e) Show that $x = 0$ is the only equilibrium of the optimal closed-loop system. Discuss the stability of this equilibrium. (Hint: study the signum of $\dot{x}$ as a function of $x$.) [4]

2. Consider the simplified model of a ship described by the equation

$$M\ddot{\theta} + d\dot{\theta} + c\alpha = w$$
$$\dot{\alpha} + \alpha = u$$

where $\theta$ denotes the heading angle error (the angle between the ship's heading and the desired heading), $\alpha$ denotes the rudder angle, $w$ denotes a disturbance due to wind, and $u$ is the control input. $M$ and $c$ are positive parameters, and $d$ is a non-negative parameter.

(a) Write the equation of the system, with state $(\theta, \dot{\theta}, \alpha)$, input $(w, u)$ and output $\theta$ in standard state space form. [4]

(b) Consider the system determined in part (a) with $w = 0$. Verify that the system is controllable. [4]

(c) Consider the system determined in part (a). Verify that the system is observable. [4]

(d) Consider the system determined in part (a) with $w = 0$. Assume $M = 1$, $c = 1$ and $d = 0$. Design an output feedback controller applying the separation principle. In particular, select the state feedback gain $K$ such that the matrix $A - BK$ has three eigenvalues equal to $-1$ and the output injection gain $L$ such that the matrix $A - LC$ has three eigenvalues equal to $-3$. [8]

3. Consider the system

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with initial state $x_0$, with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty ((x_1(t) + \beta x_2(t))^2 + u^2(t))dt$$

with $\beta \in I\!R$ and $\beta \neq 0$.

(a) Verify that the (sufficient) conditions for the existence and uniqueness of an optimal feedback control law are met. [2]

(b) Write the Riccati equation associated with this optimal control problem and find all its solutions. [4]

(c) Find the positive definite solution of the Riccati equation determined in part (b). [8]

(d) Compute the optimal control law and the optimal closed-loop system. [2]

(e) Compute the eigenvalues of the optimal closed loop system determined in part (d). Show that for $0 < |\beta| < \sqrt{2}$ the eigenvalues are complex conjugate, for $|\beta| = \sqrt{2}$ the eigenvalues are real and coincide, and for $|\beta| > \sqrt{2}$ the eigenvalues are real. Show, moreover, that as $|\beta| \to \infty$ one eigenvalues approaches zero and the other approaches $-\infty$. [4]

4. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + u \\
y &= \alpha x_1 + x_2.
\end{aligned}$$

Consider a reference signal $w(t) = [1,0]'$ and consider the problem of designing a linear static error feedback control law such that the state of the closed-loop system asymptotically tracks the signal $w$.

(a) Show that there is no linear static error feedback control law solving the considered problem. [2]

(b) Consider the system with input $y$

$$\dot{\xi} = \lambda\xi + y,$$

and set $u = \xi + v$, where $v$ is a new input signal. Write state space equations for the extended system with state $x_e = [x_1, x_2, \xi]'$ and input $v$. [4]

(c) Show that it is possible to select the parameter $\lambda$ of the extended system, determined in part (b), in a way that makes the following problem solvable: design a linear static error feedback control law such that the state of the extended closed-loop system asymptotically tracks a signal of the form $w_e = [1,0,\bar{\xi}]'$, with $\bar{\xi}$ constant. [6]

(d) Let $\lambda$ be as determined in part (d). Design a control law $u = -Kx_e + Kw_e$ which solves the asymptotic tracking problem and which is such that the eigenvalues of the closed-loop extended system are all equal to $-1$. Show that there is such a $K$ only if $\alpha = 1$. [4]

(e) Let $\lambda$ be as determined in part (c). By computing the rank of the controllability matrix of the extended system explain the results obtained in part (d). [4]

5. Consider the system
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u$$
with $u(t) \in [0, 1]$, initial state $x(0) = [x_{10}, x_{20}]'$, free final state $x(T)$, and the problem of maximizing $x_1(T)$, at some fixed time $T > 0$.

(a) Show that the considered maximization problem can be recast as the problem of minimizing
$$J = \int_0^T -x_2(t)dt.$$

[4]

(b) Write the necessary conditions of optimality for normal extremals. [4]

(c) Write the optimal control as a function of the optimal costate. [2]

(d) Consider the differential equations of the costate. Assume $\lambda_1^\star(T) = \lambda_2^\star(T) = 0$ and $\lambda_1^\star(t) = A \sin t + B \cos t + C$. Determine $\lambda_1^\star(t)$ and $\lambda_2^\star(t)$. [6]

(e) Write the optimal control as a function of time. Assume $T = 10$. Compute how many times the optimal control switches. [4]

6. Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

with initial state $x(0) = [x_{10}, x_{20}]'$, final state $x(T) = [0,0]'$, and the problem of minimizing the cost

$$J = \int_0^T u^2(t)dt$$

with fixed final time $T > 0$.

(a) Write the necessary conditions of optimality for normal extremals. (Hint: use the condition $\frac{\partial H}{\partial u} = 0$ to compute the optimal control.)                [6]

(b) Write the optimal control as a function of the optimal costate.                [2]

(c) Integrate the differential equations of the costate with initial conditions $\lambda^*(0) = [\lambda_{10}^*, \lambda_{20}^*]$.                [2]

(d) Determine the optimal control as a function of time and of $\lambda^*(0)$.                [2]

(e) Integrate the state equations with the optimal control, and use the boundary condition $x(T) = 0$ to determine $\lambda_{10}^*$ and $\lambda_{20}^*$ as a function of $T$, $x_{10}$ and $x_{20}$. Write the optimal control as a function of $T$, $x_{10}$ and $x_{20}$. Determine the initial condition $x_{10}$ and $x_{20}$ for which the corresponding optimal control is constant for all $t$.                [8]

Linear Optimal Control - Model answers 2005

# Question 1

(a) The Hamilton-Jacobi equation is (note that $\frac{\partial V}{\partial t} = 0$)

$$0 = \min_u \left[ (q(x))^2 + u^2 + \frac{\partial V}{\partial x}(f(x) + u) \right].$$

Performing the minimization yields the optimal control (as a function of $x$ and $\frac{\partial V}{\partial x}$), namely

$$u^\star = -\frac{1}{2}\frac{\partial V}{\partial x}$$

and the Hamilton-Jacobi equation

$$0 = (q(x))^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x}f(x).$$

(b) Note that the Hamilton-Jacobi equation is quadratic in $\frac{\partial V}{\partial x}$. Hence,

$$\frac{\partial V}{\partial x} = 2\left( f(x) \pm \sqrt{(f(x))^2 + (q(x))^2} \right)$$

yielding

$$V(x) = \int_0^x 2\left( f(\xi) \pm \sqrt{(f(\xi))^2 + (q(\xi))^2} \right) d\xi + c,$$

where $c$ is a constant.

(c) Setting $f(x) = -q(x)$, and $q(x) = x + x^3$, and taking the $-$ sign in the integral yields

$$V(x) = 2(\sqrt{2} - 1)\left(\frac{x^2}{2} + \frac{x^4}{4}\right) + c.$$

Setting $c = 0$ gives the desired positive definite solution.

(d) The optimal control is
$$u^\star(x) = -(\sqrt{2} - 1)\left(x + x^3\right)$$

and the optimal closed-loop system is

$$\dot{x} = f(x) + u^\star(x) = -\sqrt{2}x(1 + x^2).$$

(e) The equation $\dot{x} = 0$ has only one solution, namely $x = 0$. Note now that $\dot{x} < 0$ for $x > 0$ and $\dot{x} > 0$ for $x < 0$, hence all trajectories approach the zero equilibrium.

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# Question 2

(a) The description of the system in standard state space form is (set $x = (\theta, \dot{\theta}, \alpha)'$)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -d/M & -c/M \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1/M & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

(b) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & -c/M \\ 0 & -c/M & c/M(d/M + 1) \\ 1 & -1 & 1 \end{bmatrix}$$

and this has full rank for all positive $c$ and $M$. The system is controllable.

(c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -d/M & -c/M \end{bmatrix}$$

and this has full rank for all positive $c$ and $M$. The system is observable.

(d) Let $K = [k_1 \ k_2 \ k_3]$ and note that

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -k_1 & -k_2 & -1 - k_3 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3 + (1 + k_3)s^2 + (-k_2)s + (-k_1)$. Hence the selection

$$k_1 = -1 \qquad k_2 = -3 \qquad k_3 = 2$$

is such that the eigenvalues of $A - BK$ are equal to $-1$. Let $L = [l_1 \ l_2 \ l_3]'$ and note that

$$A - LC = \begin{bmatrix} -l_1 & 1 & 0 \\ -l_2 & 0 & -1 \\ -l_3 & 0 & -1 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3 + (1 + l_1)s^2 + (l_1 + l_2)s + (l_2 - l_3)$. Hence the selection

$$l_1 = 8 \qquad l_2 = 19 \qquad l_3 = -8$$

is such that the eigenvalues of $A - LC$ are equal to $-3$. Finally, the controller is $\dot{\xi} = (A - BK - LC)\xi + Ly$, $u = -K\xi$.

# Question 3

(a) The pair $(A, B)$ is controllable, $R = 1 > 0$, $Q = [1\ \beta]'[1\ \beta] \geq 0$, and the pair $(A, [1\ \beta])$ is observable.

(b) Set
$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

The ARE is
$$\begin{bmatrix} 1 - p_{12}^2 & p_{11} - p_{12}p_{22} + \beta \\ p_{11} - p_{12}p_{22} + \beta & 2p_{12} - p_{22}^2 + \beta^2 \end{bmatrix} = 0.$$

We obtain $p_{12} = \pm 1$. Selecting $p_{12} = 1$ yields $p_{22} = \pm\sqrt{2 + \beta^2}$ and $p_{11} = -\beta \pm \sqrt{2 + \beta^2}$. Selecting $p_{12} = -1$ yields $p_{22} = \pm\sqrt{-2 + \beta^2}$ and $p_{11} = -\beta \mp \sqrt{-2 + \beta^2}$.

(c) For any $\beta$, the only positive definite solution is
$$P = \begin{bmatrix} \sqrt{2 + \beta^2} - \beta & 1 \\ 1 & \sqrt{2 + \beta^2} \end{bmatrix}.$$

(d) The optimal control is
$$u = -[1, \ \sqrt{2 + \beta^2}]x$$
and the optimal closed-loop systems is
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2 + \beta^2} \end{bmatrix} x.$$

(e) The eigenvalues of the optimal closed-loop system are
$$\lambda_1 = -\frac{1}{2}\sqrt{\beta^2 + 2} + \frac{1}{2}\sqrt{\beta^2 - 2} \qquad \lambda_2 = -\frac{1}{2}\sqrt{\beta^2 + 2} - \frac{1}{2}\sqrt{\beta^2 - 2}.$$

Clearly, for $|\beta| < \sqrt{2}$ these eigenvalues are complex conjugate, for $|\beta| = \sqrt{2}$ they are real and coincide (equal to $-1$), and for $|\beta| > \sqrt{2}$ they are real. Finally, as $|\beta| \to \infty$, we have $\lambda_1 \to 0$ and $\lambda_2 \to -\infty$.

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# Question 4

(a) To achieve asymptotic tracking with the stated class of feedback, it is necessary that $\dot{w} - Aw = 0$. In particular this is not the case for the given signal. In fact, if $w(t) = [1,0]'$ then $\dot{w}(t) = [0,0]'$ and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq 0.$$

(b) The state space equations for the extended system are

$$\dot{x}_e = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ \alpha & 1 & \lambda \end{bmatrix} x_e + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v = A_e x_e + B_e v$$

(c) Note now that $w_e = [1,0,\bar{\xi}]'$, $\dot{w}_e = [0,0,0]'$ and

$$\dot{w}_e - A_e w_e = \begin{bmatrix} 0 \\ 1 - \bar{\xi} \\ -\alpha - \lambda\bar{\xi} \end{bmatrix}.$$

Hence, selecting $\bar{\xi} = 1$ and $\lambda = -\alpha$ we satisfy the condition for the existence of a static error feedback achieving asymptotic tracking.

(d) Let $K = [k_1, k_2, k_3]$ and note that

$$A_e - B_e K = \begin{bmatrix} 0 & 1 & 0 \\ -1 - k_1 & -k_2 & 1 - k_3 \\ \alpha & 1 & -\alpha \end{bmatrix}.$$

The characteristic polynomial of the above matrix is

$$s^3 + (\alpha + k_2)s^2 + (k_1 + k_3 + \alpha k_2)s + \alpha(k_1 + k_3),$$

and this should be equal to $(s+1)^3$, *i.e.* we have to solve the equations

$$\alpha + k_2 = 3 \qquad k_1 + k_3 + \alpha k_2 = 3 \qquad \alpha(k_1 + k_3) = 1.$$

Note that these equations are not independent. From the first one we obtain $k_2 = 3 - \alpha$. Substituting in the second one we have $k_1 + k_3 = 3 - \alpha(3 - \alpha)$ however, from the third equation we obtain $k_1 + k_3 = \frac{1}{\alpha}$. Therefore

$$\frac{1}{\alpha} = 3 - \alpha(3 - \alpha).$$

Plotting the left-hand side and the right-hand side we see that this equation has only one solution: $\alpha = 1$. Hence, there is a $K$ solving the stated problem problem only if $\alpha = 1$, namely

$$K = [1 - k_3, 2, k_3].$$

(e) The controllability matrix, for $\lambda = -\alpha$, is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and this has rank 2. It is therefore not possible to assign arbitrarily the eigenvalues of $A_e - B_e K$. Note however, that for $\alpha = 1$ the uncontrollable mode is $s = -1$, hence there is a feedback gain assigning the desired eigenvalues.

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# Question 5

(a) From the $\dot{x}_1$ equation we have

$$x_1(T) = x_1(0) + \int_0^T x_2(t)dt,$$

hence, maximizing $x_1(T)$ is equivalent to maximizing

$$\int_0^T x_2(t)dt,$$

which is equivalent to minimizing

$$\int_0^T -x_2(t)dt.$$

(b) Let

$$H = -x_2 + \lambda_1 x_2 + \lambda_2(-x_1 + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1 + u$$

$$\dot{\lambda}_1 = \lambda_2 \qquad \dot{\lambda}_2 = 1 - \lambda_1$$

$$\lambda_2 u \leq \lambda_2 \omega, \quad \forall \omega \in [0,1].$$

(c) The optimal control as a function of the costate is

$$u^\star(t) = \begin{cases} 0 & \text{if } \lambda_2^\star(t) > 0 \\ 1 & \text{if } \lambda_2^\star(t) < 0 \end{cases}$$

If $\lambda_2^\star(t) = 0$ we do not have information on the optimal control.

(d) From $\lambda_1^\star = A\sin t + B\cos t + C$, we have $\lambda_2^\star = A\cos t - B\sin t$. Setting $\lambda_1(T) = \lambda_2(T) = 0$ and solving for $A$, $B$ and $C$ yields

$$\lambda_1^\star = 1 - \cos(t - T) \qquad \lambda_2^\star = \sin(t - T).$$

(e) The optimal control as a function of time is

$$u^\star(t) = \begin{cases} 0 & \text{if } \sin(t - T) > 0 \\ 1 & \text{if } \sin(t - T) < 0 \end{cases}$$

Hence, for $T = 10$ the optimal control switches 3 times.

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# Question 6

(a) Let
$$H = u^2 + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality, for normal extremals, are
$$\dot{x}_1 = x_2 \quad \dot{x}_2 = u$$
$$\dot{\lambda}_1 = 0 \quad \dot{\lambda}_2 = -\lambda_1$$
$$2u + \lambda_2 = 0.$$

(b) The optimal control as a function of the costate is
$$u^\star = -\frac{1}{2}(\lambda_2^\star(t)).$$

(c) From the necessary conditions in part (a) we obtain
$$\lambda_1^\star(t) = \lambda_{10}^\star$$
$$\lambda_2^\star(t) = \lambda_{20}^\star - \lambda_{10}^\star t.$$

(d) The optimal control is
$$u^\star = -\frac{1}{2}(\lambda_{20}^\star - \lambda_{10}^\star t).$$

(e) Integrating the state equations with the optimal control we obtain
$$x_1^\star(t) = x_{10} + x_{20}t - \frac{1}{4}\lambda_{20}^\star t^2 + \frac{1}{12}\lambda_{10}^\star t^3$$

$$x_2^\star(t) = x_{20} - \frac{1}{2}\lambda_{20}^\star t + \frac{1}{4}\lambda_{10}^\star t^2.$$

Setting $x_1(T) = x_2(T) = 0$ and solving for $\lambda_{10}^\star$ and $\lambda_{20}^\star$ yields
$$\lambda_{10}^\star = 12\frac{x_{20}T + 2x_{10}}{T^3} \qquad \lambda_{20}^\star = -4\frac{2x_{20}T + 3x_{10}}{T^2}.$$

As a result, the optimal control is
$$u^\star(t) = -2\frac{3x_{10} + 2x_{20}T}{T^2} + 6\frac{x_{20}T + 2x_{10}}{T^3}t.$$

This is constant for all initial states such that
$$x_{20}T + 2x_{10} = 0.$$

Finally, for such initial conditions the optimal control is $u^\star(t) = -\frac{x_{20}}{T}$.