E4.22
C1.2

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE PART IV: M.Eng. and ACGI

LINEAR OPTIMAL CONTROL

Monday, 26 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	G. Weiss

Special instructions for invigilators: None

Information for candidates:

Hamilton Jacobi equation:

$$\dot{x}(t) = f(t, x, u),\ x(0) = x_0$$

$$J(x_0, u) = \int_{\tau}^{T} L(t, x, u)dt + m(x(T)),$$

$$-\frac{\partial V}{\partial t} = \min_u \left[L(t, x, u) + \frac{\partial V}{\partial x} f(t, x, u) \right],\ V(x, T) = m(x)$$

Linear Quadratic Regulator:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0$$

$$J(x_0, u) = \int_{\tau}^{T} \left[x(t)'Qx(t) + u(t)'Ru(t) \right] dt + x(T)'Mx(T)$$

$$Q = Q' \geq 0,\ R = R' > 0,\ M = M' \geq 0$$

$$-\dot{P} = A'P + PA + Q - PBR^{-1}B'P,\ P(T) = M$$

$$u(t) = -R^{-1}B'Px(t) = -Kx(t).$$

The matrices A, B, Q, R, P and K may depend upon t.

Minimum principle:

$$\dot{x} = f(x, u),\ u \in \mathcal{U}$$

$$J(x_0, u) = \int_0^{t_f} L(x(t), u(t))dt,$$

$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda' f(x, u),$$

$$\dot{\lambda}^\star = -\left.\frac{\partial H}{\partial x}\right|'_{(x^\star, u^\star, \lambda_0^\star, \lambda^\star)},$$

$$H(x^\star, \omega, \lambda_0^\star, \lambda^\star) \geq H(x^\star, u^\star, \lambda_0^\star, \lambda^\star),\ \forall \omega \in \mathcal{U},$$

$$H(x^\star, u^\star, \lambda_0^\star, \lambda^\star) = k.$$

[4.22]

1. Consider the system

$$\dot{x} = xu$$

with initial state $x_0 > 0$ and with the cost to be minimised

$$J(x_0, u) = \int_{\tau}^{T} L(x, u)dt$$

where

$$L(x, u) = (\log|x|)^2 + u^2.$$

(a) Write the Hamilton-Jacobi equation associated with this optimal control problem and the corresponding boundary condition. [4]

(b) Solve the Hamilton-Jacobi equation derived in part (a). (Hint: you may consider a *separable* solution: $V(x,t) = \alpha(t)W(x)$, with $W(x) = (\log|x|)^2$. Also, recall that $\int \frac{1}{1-x^2}dx = \text{arctanh}(x)$.) [8]

(c) Compute the optimal control and the optimal closed-loop system. [4]

(d) Suppose $(T - \tau) \to +\infty$. Compute the steady state solution of the Hamilton-Jacobi equation, the corresponding steady state optimal control and optimal closed-loop system. Show that the this closed-loop system has (for $x > 0$) the unique equilibrium $x = 1$, and discuss the stability of this equilibrium. (Hint: study the signum of $\dot{x}$ as a function of x.) [4]

[4.22]

2. Consider the system

$$\dot{x} = u$$

with initial state x_0 and with the quadratic cost to be minimised

$$J(x_0, u) = \int_{\tau}^{T} L(x, u)dt + Mx^2(T)$$

where

$$L(x, u) = u^2$$

and $M \geq 0$.

(a) Write the differential Riccati equation associated with the considered optimal control problem and the corresponding boundary condition. [4]

(b) Solve the differential Riccati equation derived in part (a). Verify that the solution is positive for all $T \geq t$. [6]

(c) Compute the optimal control and the optimal closed-loop system. [2]

(d) Suppose that $(T - \tau) \rightarrow +\infty$, $M = 0$ and compute the steady state solution of the differential Riccati equation. Show that this is equal to the solution of the algebraic Riccati equation. Compute the steady state optimal control law and the corresponding closed-loop system. Discuss the stability property of this closed-loop system and explain why the steady state optimal control law is not stabilizing. [8]

[4.22]

3. Consider the system

$$\dot{x} = Ax + Bu \quad = \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad + \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with initial state x_0, with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty (q_{11} x_1^2(t) + u^2(t))dt$$

with $q_{11} > 0$.

(a) Verify that, for any $q_{11} > 0$, the conditions for the existence and uniqueness of an optimal feedback control law are met. [2]

(b) Write the Hamiltonian matrix H associated with this optimal control problem. Show that the characteristic polynomial of H is $p(s) = s^4 + q_{11}$ and compute the eigenvalues of H as a function of q_{11}. [7]

(c) Let $u = -Kx = -k_1x_1 - k_2x_2$. Find k_1 and k_2 such that the eigenvalues of the resulting closed loop-system coincide with the eigenvalues of H having negative real part. [4]

(d) Explain why the state feedback control law constructed in part (c) solves the above optimal control problem. [4]

(e) Use the result in part (d) to compute the optimal cost associated with the initial state $x(0) = [0, x_{20}]'$. [3]

[4.22]

4. Consider the system

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u.$$

Consider a reference signal $w(t) = [w_1(t), w_2(t), w_3(t)]'$ and consider the problem of designing a linear time-invariant error feedback control law such that the state of the closed-loop system asymptotically tracks the signal w.

(a) Characterize the class of reference signals for which the above asymptotic tracking problem is solvable, and show that $w(t) = [\sin t, \cos t, 0]'$ belongs to this class. [4]

(b) Suppose $w(t)$ is such that the asymptotic tracking problem is solvable. Design a control law $u = -Kx + Kw$ which solves the asymptotic tracking problem and which is such that the eigenvalues of the closed-loop matrix $(A - BK)$ are all equal to -1. Note that there are infinitely many matrices K assigning the eigenvalues of the closed-loop system as required and discuss why this is the case. [12]

(c) Suppose $w(t)$ does not belong to the class of signals introduced in part (a). Explain how the tracking problem can be reformulated as a disturbance attenuation problem. [4]

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5. A linear system is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= \alpha x_1 + u \\ y &= x_1 - x_2 \end{aligned}$$

where $u \in I\!R$ is the control input, y is the output variable and α is a constant parameter.

(a) Study the controllability and stabilizability properties of the system as a function of α. [4]

(b) Study the observability and detectability properties of the system as a function of α. [4]

(c) Assume $\alpha \neq 1$. Design an output feedback controller applying the separation principle. In particular, select the state feedback gain K such that the matrix $(A - BK)$ has two eigenvalues equal to -1 and the output injection gain L such that the matrix $(A - LC)$ has two eigenvalues equal to -3. Note that K and L may depend on α. [8]

(d) Compute

$$\lim_{\alpha \to 1} \|K\| \qquad \lim_{\alpha \to 1} \|L\|$$

and explain your results using the solutions of parts (a) and (b). [4]

[4.22]

6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= u \end{aligned}$$

with $u \in [-1,1]$, initial state $x(0) = [x_{10}, x_{20}, \cdots, x_{n0}]'$, final state $x(T) = [0, 0, \cdots, 0]'$, and the cost (to be minimized)

$$J(x_0, u) = \int_0^T 1 \; dt.$$

(a) Write the necessary conditions of optimality for normal extremals. [4]

(b) Compute the optimal control as a function of the costate $\lambda = [\lambda_1, \lambda_2, \cdots, \lambda_n]'$ and show that $|u^\star(t)| = 1$ for all t such that $\lambda_n^\star(t) \neq 0$. [4]

(c) Use the differential equations of the costate to show that the optimal control law has at most $n-1$ switches. (Hint: show that the equation $\lambda_n^\star(t) = 0$ has at most $n-1$ solutions.) [6]

(d) Assume $n = 2$, *i.e.* consider the system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$. Integrate the state equation with $u = 1$. Determine the set of initial conditions for which the control $u(t) = 1$ for all $t \in [0, T]$ is optimal. For such initial conditions compute the time to reach the origin. [6]

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Linear Optimal Control - Model answers 2004

Question 1

(a) The Hamilton-Jacobi equation is

$$-\frac{\partial V}{\partial t} = \min_u \left[(\log|x|)^2 + u^2 + \frac{\partial V}{\partial x} x u \right].$$

Performing the minimization yields the optimal control (as a function of x and $\frac{\partial V}{\partial x}$), namely

$$u^\star = -\frac{1}{2}\frac{\partial V}{\partial x} x$$

and the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t} = (\log|x|)^2 - \frac{1}{4}\left(\frac{\partial V}{\partial x} x\right)^2,$$

with boundary condition $V(x,T) = 0$.

(b) Let $V(x,t) = \alpha(t)(\log|x|)^2$ and note that

$$\frac{\partial V}{\partial t} = \dot{\alpha}(t)(\log|x|)^2 \qquad \frac{\partial V}{\partial x} = \alpha(t)2(\log|x|)\frac{1}{x}.$$

Hence, the Hamilton Jacobi equation becomes

$$-\dot{\alpha}(t)(\log|x|)^2 = (1 - \alpha^2(t))(\log|x|)^2.$$

To obtain a solution of the above equation we simply need to integrate the ordinary differential equation $-\dot{\alpha}(t) = 1 - \alpha^2(t)$ with boundary condition $\alpha(T) = 0$. Integrating the differential equation by separation of variables, and taking into account the boudary condition we obtain

$$\operatorname{arctanh}(\alpha(t)) = -(t - T) = T - t$$

hence

$$\alpha(t) = \tanh(T - t).$$

(c) The optimal control is

$$u^\star = -\frac{1}{2}\frac{\partial V}{\partial x} x = -\tanh(T-t)(\log|x|)$$

and the optimal closed-loop system is

$$\dot{x} = -\tanh(T-t)(\log|x|)x.$$

(d) For $T - t \to \infty$ one has $\alpha(t) \to 1$, hence the steady state solution of the Hamilton Jacobi equation is $V(x) = (\log|x|)^2$ and the resulting closed-loop system is $\dot{x} = -(\log|x|)x$. This system, for $x > 0$, has the equilibrium $x = 1$, and this is asymptotically stable, in fact $\dot{x} < 0$ for $x > 1$ and $\dot{x} > 0$ for $x \in (0,1)$.

Alternatively, one may define a new variable $z = \log|x|$ and note that the problem becomes a linear quadratic problem.

Question 2

(a) Note that $A = 0$, $B = 1$, $R = 1$ and $Q = 0$. Hence, the differential Riccati equation is

$$-\dot{P} = -P^2$$

with boundary condition $P(T) = M$.

(b) Solving the differential Riccati equation by separation of variables, taking into account the boundary condition, yields

$$\frac{1}{P(t)} = T - t + 1/M$$

hence

$$P(t) = \frac{M}{M(T-t)+1}$$

which is positive for all $T \geq t$.

(c) The optimal control is

$$u^\star(t) = -\frac{M}{M(T-t)+1}x(t)$$

and the optimal closed-loop system is

$$\dot{x}(t) = -\frac{M}{M(T-t)+1}x(t).$$

(d) For $T - \tau \to \infty$ and $M = 0$ we have $P \to 0$. Note that for this problem the algebraic Riccati equation is $0 = -P^2$, with solution $P = 0$, and this coincides with the limit solution of the differential Riccati equation. The steady state optimal control is $u^\star = 0$ and the optimal closed-loop system is $\dot{x} = 0$. This is not asymptotically stable. This is due to the fact that the linear quadratic regulator problem described by $\min_u \int_0^\infty u^2 dt$ with $\dot{x} = u$, does not satisfy all the standard assumptions for the solvability of the problem, namely the matrix Q is zero, and hence the pair $(A, Q^{1/2})$ is not observable neither detectable.

Question 3

(a) The pair $\{A, B\}$ is controllable, $R = 1 > 0$, $Q = \text{diag}(q_{11}, 0) \geq 0$, and the pair $\{A, Q^{1/2}\}$ is observable.

(b) The Hamiltonian matrix is

$$H = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -q_{11} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right].$$

To obtain the characteristic polynomial $p(s)$ of H compute the determinant of $sI - H$ using the 'expansion by minors method' starting from the first row. This yields $p(s) = s(s^3) + 1(q_{11})$. The eigenvalues of H are the roots of $p(s)$, namely

$$(q_{11})^{1/4}\left(\pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right).$$

(c) Note that

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^2 + k_2 s + k_1$. This should be equal to

$$\left(s - (q_{11})^{1/4}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\right)\left(s - (q_{11})^{1/4}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})\right) = s^2 + \sqrt{2}(q_{11})^{1/4}s + \sqrt{q_{11}}.$$

Hence

$$k_1 = \sqrt{q_{11}}, \qquad k_2 = \sqrt{2}(q_{11})^{1/4}.$$

(d) From the general theory of the linear quadratic regulator we know that the eigenvalues of the optimal closed-loop system coincide with the stable eigenvalues of the Hamiltonian matrix of the problem. However, because the system considered has only one input (and it is controllable), given any two complex numbers λ_1 and λ_2 there is only one K such that $\lambda(A - BK) = \{\lambda_1, \lambda_2\}$. Therefore, the K obtained assigning to the closed-loop system the stable eigenvalues of the Hamiltonian matrix is the optimal K.

(e) The optimal cost is $x(0)'Px(0)$, and for $x(0) = [0, x_{20}]'$ it is $x_{20}^2 P_{22}$. However, the optimal feedback gain is $K = R^{-1}B'P = [P_{12}, P_{22}]$. Therefore $P_{22} = \sqrt{2}(q_{11})^{1/4}$ and the optimal cost for the initial state $[0\ x_{20}]'$ is $\sqrt{2}(q_{11})^{1/4}x_{20}^2$.

Question 4

(a) To achieve asymptotic tracking with the stated class of feedback, it is necessary that $\dot{w} - Aw = 0$. In particular this is the case for the given signal. In fact, if $w(t) = [\sin t, \cos t, 0]'$ then $\dot{w}(t) = [\cos t, -\sin t, 0]'$ and

$$\begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix} = 0.$$

(b) Let $e = x - w$ and note that

$$\dot{e} = \dot{x} - \dot{w} = Ax + Bu - Aw = Ae + Bu.$$

Let now

$$u = -Ke = -K(x - w) = -Kx + Kw$$

and select $K = [k_1, k_2, k_3]$ such that the eigenvalues of $A - BK$ are all equal to -1. Note that

$$A - BK = \begin{bmatrix} -k_1 & 1 - k_2 & 1 - k_3 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix},$$

the characteristic polynomial of $A - BK$ is

$$p(s) = (s + 1)(s^2 + k_1 s + 1 - k_2)$$

and this should be equal to $(s + 1)^3$. This is achieved by selecting $k_1 = 2$ and $k_2 = 0$, while k_3 can be arbitrarily assigned. This is due to the fact that the pair (A, B) is not controllable, however it is stabilizable and the uncontrollable mode is $s = -1$. This can be seen considering the controllability pencil

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & -1 & -1 & 1 \\ 1 & s & -2 & 0 \\ 0 & 0 & s+1 & 0 \end{array}\right]$$

and noting that it has rank 2 for $s = -1$ and rank 3 for any other s.

(c) If $\dot{w} \neq Aw$ then we can still consider the error variable $e = x - w$ and define a disturbance $d = Aw - \dot{w}$. As a result

$$\dot{e} = Ae + Bu + d$$

and the tracking problem, *i.e.* the problem of rendering e small can be regarding as a disturbance attenuation problem, *i.e.* the problem of finding $u = -Ke$ such that the effect of d on e is small.

Question 5

(a) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & \alpha \end{bmatrix}$$

and the system is controllable if $\alpha \neq 1$. If $\alpha = 1$ the controllability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s & -1 & 1 \\ -1 & s & 1 \end{array}\right]$$

and this has rank 1 for $s = -1$ and rank 2 for any other s. Therefore, the system is stabilizable and $s = -1$ is the non-controllable mode.

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & -1 \\ -\alpha & 1 \end{bmatrix}$$

and the system is observable if $\alpha \neq 1$. If $\alpha = 1$ the observability pencil is

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s & -1 \\ -1 & s \\ \hline 1 & -1 \end{array}\right]$$

and this has rank 1 for $s = 1$ and rank 2 for any other s. Therefore the system is not detectable.

(c) Let $K = [k_1\ k_2]$ and note that

$$A - BK = \begin{bmatrix} -k_1 & 1 - k_2 \\ \alpha - k_1 & -k_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^2 + (k_1 + k_2)s + (k_1 + \alpha k_2 - \alpha)$. Hence the selection

$$k_1 = 1 \qquad k_2 = 1$$

is such that the eigenvalues of $A - BK$ are equal to -1. Let $L = [l_1\ l_2]^T$ and note that

$$A - LC = \begin{bmatrix} -l_1 & 1 + l_1 \\ \alpha - l_2 & l_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^2 + (l_1 - l_2)s + (l_2 - \alpha l_1 - \alpha)$. Hence the selection

$$l_1 = \frac{15 + \alpha}{1 - \alpha} \qquad l_2 = \frac{9 + 7\alpha}{1 - \alpha}$$

is such that the eigenvalues of $A - LC$ are equal to -3. Finally, the controller is $\dot{\xi} = (A - BK - LC)\xi + Ly$, $u = -K\xi$.

(d) The limit for $\alpha \to 1$ of $\|K\| = \sqrt{k_1^2 + k_2^2}$ is $\sqrt{2}$, whereas the limit for $\alpha \to 1$ of $\|L\|$ is equal to $+\infty$. This is in agreement with the fact that, for $\alpha = 1$, the system is not controllable but stabilizable, with a non-controllable mode equal to -1, but not detectable.

Question 6

(a) Let

$$H = 1 + \lambda_1 x_2 + \lambda_2 x_3 + \cdots \lambda_n u.$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_3 \quad \cdots \dot{x}_n = u$$

$$\dot{\lambda}_1 = 0 \quad \dot{\lambda}_2 = -\lambda_1 \quad \cdots \dot{\lambda}_n = -\lambda_{n-1}$$

$$\lambda_n u \leq \lambda_n \omega, \ \forall \omega \in [-1, 1].$$

(b) The optimal control as a function of λ_n is

$$u^\star = -\text{sign}(\lambda_n^\star(t)).$$

Hence, for any t such that $\lambda_n^\star(t) \neq 0$, $|u^\star(t)| = 1$.

(c) From the necessary conditions in part (a) we obtain

$$\begin{aligned}
\lambda_1^\star(t) &= \lambda_1^\star(0) \\
\lambda_2^\star(t) &= \lambda_2^\star(0) - \lambda_1^\star(0)t \\
\lambda_3^\star(t) &= \lambda_3^\star(0) - \lambda_2^\star(0)t + \lambda_1^\star(0)\tfrac{t^2}{2} \\
&\vdots
\end{aligned}$$

Hence, $\lambda_n^\star(t)$ is a polynomial in t of degree $n-1$. Therefore the equation

$$\lambda_n^\star(t) = 0$$

in the unknwon t may have at most $n-1$ solutions, say $\bar{t}_1$, $\bar{t}_2$, $\cdots$, $\bar{t}_{n-1}$. This means that if $\bar{t}_i \in (0, T)$ then the optimal control will have a *switch* either from $+1$ to -1 or from -1 to $+1$ at $t = \bar{t}_i$, and this may occur at most $n-1$ times.

(d) If $\dot{x}_1 = x_2$ and $\dot{x}_2 = u$ and $u = 1$ then

$$x_2(t) = x_{20} + t \qquad x_1(t) = x_{10} + x_{20}t + \frac{t^2}{2}$$

These trajectories are optimal if they are such that $x_1(T) = x_2(T) = 0$, *i.e.*

$$0 = x_{20} + T = x_{10} + x_{20}T + \frac{T^2}{2}$$

Eliminating the variable $T > 0$ we obtain $T = -x_{20} > 0$ hence

$$x_{10} - \frac{x_{20}^2}{2} = 0.$$

Hence the set of initial conditions which are driven to the origin at time T by the control $u = 1$ consists of those $[x_{10}, x_{20}]^T$ for which

$$x_{10} - \frac{x_{20}^2}{2} = 0$$

with $x_{20} \leq 0$, and the time to reach the origin is $T = -x_{20} \geq 0$.