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C1.2

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE PART IV: M.Eng. and ACGI

LINEAR OPTIMAL CONTROL

Monday, 22 April 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners responsible:

First Marker(s): Astolfi,A.

Second Marker(s): Weiss,G.

Special instructions for invigilators: None

Information for candidates:

System: $\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0.$

Quadratic cost function: $J(x_0, u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)]\, dt,$
$Q = Q' \geq 0,\ R = R' > 0.$

Riccati equation: $A'P + PA + Q - PBR^{-1}B'P = 0.$

Optimal control law: $u(t) = -R^{-1}B'Px(t) = -Kx(t).$

Minimum cost: $x_0'Px_0.$

Return difference inequality for scalar u: $|1 + K(j\omega I - A)^{-1}B| \geq 1,$

Minimum principle:

$$\dot{x} = f(x, u),\ u \in \mathcal{U}$$

$$J(x_0, u) = \int_0^{t_f} L(x(t), u(t))dt,$$

$$H(x, u, \lambda_0, \lambda) = \lambda_0 L(x, u) + \lambda^T f(x, u),$$

$$\dot{\lambda}^\star = -\frac{\partial H}{\partial x}\bigg|^T_{(x^\star, u^\star, \lambda_0^\star, \lambda^\star)},$$

$$H(x^\star, \omega, \lambda_0^\star, \lambda^\star) \geq H(x^\star, u^\star, \lambda_0^\star, \lambda^\star),\ \forall \omega \in \mathcal{U},$$

$$H(x^\star, u^\star, \lambda_0^\star, \lambda^\star) = k.$$

1. Consider the linear electric network in Figure 1, with $R_1 > 0$, $R_2 > 0$, $C > 0$ and $L > 0$. Denote by u be the driving voltage, by x_1 the current through the inductor L, by x_2 the voltage across the capacitor C, and by y the current through the voltage source.

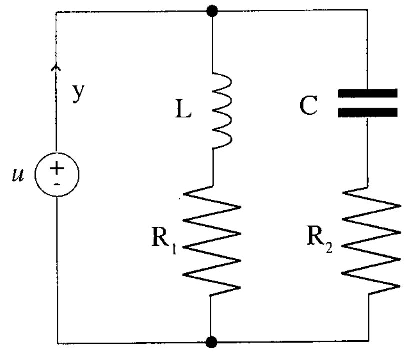

Figure 1.

(a) Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in the standard state-space form, regarding u as the input and y as the output. [6]

(b) Study the controllability and the observability of the dynamical system determined in part (a). [6]

(c) Compute the transfer function from the input u to the output y. [4]

(d) Compute values of R_1, R_2, C and L such that in the transfer function computed in part (c) there is a pole-zero cancellation. Interpret your answer in the light of your answer to part (b). [4]

2. A linear system is described by the differential equations

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \alpha x_3 + u_1\\ \dot{x}_3 &= \beta x_1 + \gamma x_2 + u_2\end{aligned}$$

where $u_1 \in I\!R$ and $u_2 \in I\!R$ are control inputs and α, β and γ are constant parameters.

(a) Suppose that only u_1 is used for control (i.e. $u_2 = 0$). Study the controllability and stabilizability properties of the system as a function of α, β and γ. [4]

(b) Suppose that both u_1 and u_2 can be used for control. Show that the system is controllable for any α, β and γ. [7]

(c) Set $\alpha = 1$, $\beta = 0$ and $\gamma = -1$. Let $u_1 = k_1x_1 + k_3x_3$ and $u_2 = f_1x_1 + f_3x_3$. Find values of k_1, k_3, f_1 and f_3 such that the closed-loop system has all three eigenvalues at $-\lambda^\star$, with $\lambda^\star > 0$. Show that such values of k_1, k_3, f_1 and f_3 are not unique. Compute the values of k_1, k_3 and f_3, which together with $f_1 = 0$, assign all the eigenvalues of the closed-loop system at $-\lambda^\star$. [9]

3. Consider the system

$$\dot{x} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

with initial state x_0, with the quadratic cost to be minimised

$$J(x_0, u) = \int_0^\infty \left(q_{11}x_1^2 + q_{22}x_2^2 + ru^2 \right) dt,$$

with $x = [x_1, x_2]'$, $q_{11} > 0$, $q_{22} > 0$ and $r > 0$.

(a) Verify that the conditions for the existence and uniqueness of an optimal feedback control law are met. [4]

(b) Write the ARE associated with the above optimal control problem. Find q_{11}, q_{22} and p_{22} such that the ARE is satisfied by a matrix of the form

$$P = \begin{bmatrix} 2 & 0 \\ 0 & p_{22} \end{bmatrix}.$$

Verify that the resulting Q and P are positive definite for all $r \in (0, 1)$. [8]

(c) For q_{11} and q_{22} as determined in part (b) compute the optimal feedback law. Show that the eigenvalues of the optimal closed-loop system go to -3 and to $-\infty$ as $r \to 0$. (Hint: Re-write the optimal closed-loop system in the new state variable $z = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x$.) [8]

4. Consider the system

$$\dot{x} = x^3 + u,$$

the initial state $x(0) = x_0 \neq 0$, the final state $x(T) = 0$ and the cost (to be minimized)

$$J(x_0, u) = \int_0^T \frac{1}{2} u^2 dt.$$

(a) Write the necessary conditions of optimality in the case of normal extremals. (Note that there is no constraint on u.) [6]

(b) Compute the optimal control as a function of the costate λ. Then, using the condition that the Hamiltonian is zero along any extremal, compute the optimal control as a function of x. [6]

(c) Integrate the state equation and find the optimal control law as a function of t. Finally, compute the time T at which the condition $x(T)$ is met, and compute the optimal cost for any x_0. [8]

5. Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

with $A + A^T = 0$ and $C = B^T$, and the quadratic cost (to be minimised)

$$J(x_0, u) = \int_0^{\infty} \left(\alpha y^T(t)y(t) + u^T(t)u(t)\right) dt,$$

with $\alpha > 0$.

(a) Show that the system is controllable if and only if it is observable. [4]

(b) Write the ARE associated with the above optimal control problem. [2]

(c) Find the positive definite solution P of the ARE derived in part (b) and compute the optimal state feedback control law and the optimal closed-loop system. (Hint: consider a diagonal P.) [6]

(d) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad B = C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the results in part (c), compute the optimal closed-loop system as a function of α. Study the behaviour of the eigenvalues of the optimal closed-loop system when α goes to infinity. Compute the transfer function $G(s)$ of the system (A, B, C) and verify that, as $\alpha \to \infty$, one eigenvalue of the closed-loop system approches the zero of $G(s)$. [8]

6. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\end{aligned}$$

and the problem of finding a bounded control law $|u(t)| \leq 1$ that drives the state of the system from $(x_1(0), x_2(0)) = (-1/2, 1)$ to $(x_1(t_f), x_2(t_f)) = (0, 0)$ in minimum time.

(a) Write the necessary conditions of optimality for normal extremals. [4]

(b) Write the optimal control as a function of the optimal costate $\lambda^\star(t)$. [2]

(c) Assume that the optimal control has constant sign in the interval $t \in [0, t_f]$, integrate the state equations and compute the optimal control $u^\star(t)$ as a function of t. Thus evaluate t_f. [6]

(d) Integrate the costate equations and verify that there exists an initial condition $\lambda(0)$ for the costate such that the Hamiltonian is equal to zero along the optimal solution computed in part (c). [4]

(e) Show that the control that drives the state of the system from $(x_1(0), x_2(0)) = (1/2, -1)$ to $(x_1(t_f), x_2(t_f)) = (0, 0)$ in minimum time is $-u^\star(t)$, where $u^\star(t)$ is the control computed in part (c).
(Hint: Define a new state variable $z = -x$.) [4]

Linear Optimal Control - Model answers 2002

Question 1

(a) Let i_1 and i_2 be the currents through R_1 and R_2, respectively. Then $y = i = i_1 + i_2$. Moreover,

$$i_1 = x_1 = \frac{u - L\dot{x}_1}{R_1} \qquad i_2 = C\dot{x}_2 = \frac{u - x_2}{R_2}.$$

Hence,

$$A = \begin{bmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2 C} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_2 C} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -\frac{1}{R_2} \end{bmatrix} \qquad D = \frac{1}{R_2}.$$

(b) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} \frac{1}{L} & -\frac{R_1}{L^2} \\ \frac{1}{R_2 C} & -\frac{1}{R_2^2 C^2} \end{bmatrix}$$

and it is full rank if $R_1 R_2 C \neq L$. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & -\frac{1}{R_2} \\ -\frac{R_1}{L} & \frac{1}{R_2^2 C} \end{bmatrix}$$

and it is full rank if $R_1 R_2 C \neq L$.

(c) The transfer function is

$$W(s) = C(sI - A)^{-1}B + D = \frac{s^2 LC + sC(R_1 + R_2) + 1}{(Ls + R_1)(CR_2 s + 1)}.$$

(d) There is a pole-zero cancellation if

$$(s^2 LC + sC(R_1 + R_2) + 1)_{s=-R_1/L} = 0$$

or

$$(s^2 LC + sC(R_1 + R_2) + 1)_{s=-1/(CR_2)} = 0$$

and this is the case if $R_1 R_2 C = L$. This is expected, in fact, from part (b) we know that if $R_1 R_2 C = L$ then the system is neither controllable nor observable.

Question 2

(a) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha\gamma \\ 0 & \gamma & \beta \end{bmatrix}$$

and it is full rank if $\beta \neq 0$. Hence the system is controllable if and only if $\beta \neq 0$. To check stabilizability, consider the matrix

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \gamma & 1 \end{bmatrix}.$$

Then, with $\beta = 0$,

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & \alpha\gamma & \alpha \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

showing that the system is not stabilizable (the un-controllable mode has eigenvalue equal to 0).

(b) The controllability matrix is now composed of six columns ($\star$ denotes an element which does not need to be computed)

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 & \star & \star \\ 1 & 0 & 0 & \alpha & \star & \star \\ 0 & 1 & \gamma & 0 & \star & \star \end{bmatrix}.$$

This matrix has always rank three, as the first three columns are linearly independent, for any α, β and γ. The system is controllable for any α, β and γ.

(c) The closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} 0 & 1 & 0 \\ k_1 & 0 & k_3 + 1 \\ f_1 & -1 & f_3 \end{bmatrix} x$$

and the characteristic polynomial of A_{cl} is

$$\lambda^3 - f_3\lambda^2 + (k_3 + 1 - k_1)\lambda + (k_1 f_3 - f_1 - f_1 k_3).$$

This has to be equal to $(\lambda + \lambda^\star)^3$. Hence, equating coefficients with equal power yields

$$f_3 = -3\lambda^\star \qquad k_3 + 1 - k_1 = 3(\lambda^\star)^2 \qquad k_1 f_3 - f_1 - f_1 k_3 = (\lambda^\star)^3$$

and this system of equations admits infinitely many solutions k_1, k_3, f_1 and f_3. However, if $f_1 = 0$ then the only solution is

$$k_1 = -\frac{(\lambda^\star)^2}{3} \qquad k_3 = 3(\lambda^\star)^2 - 1 - \frac{(\lambda^\star)^2}{3} \qquad f_3 = -3\lambda^\star.$$

Question 3

(a) The pair (A, B) is controllable, the pair $(A, Q^{1/2})$, with $Q = \text{diag}(q_{11}, q_{22})$, is observable, $Q > 0$ and $R = r > 0$.

(b)

$$A'P + PA - PBR^{-1}B'P + Q = \begin{bmatrix} 4 - \frac{4}{r} + q_{11} & 3p_{22} - 4 + 2\frac{p_{22}}{r} \\ 3p_{22} - 4 + 2\frac{p_{22}}{r} & q_{22} - 2p_{22} - \frac{p_{22}^2}{r} \end{bmatrix} = 0.$$

The above equation has the solution

$$q_{11} = 4\frac{1-r}{r} \qquad q_{22} = \frac{8r(3r+4)}{9r^2+12r+4} \qquad p_{22} = \frac{4r}{3r+2}.$$

Note that $P > 0$ for any $r > 0$ and $Q > 0$ for any $r \in (0, 1)$.

(c) The optimal feedback is $u = -kx$ with

$$k = \frac{1}{r}[1 \quad -1]\text{diag}(2, \frac{4r}{3r+2})$$

and the optimal closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} 1 - \frac{2}{r} & -2 + \frac{4}{3r+2} \\ 3 + \frac{2}{r} & -1 - \frac{4}{3r+2} \end{bmatrix} x.$$

Note that in the variable z one has

$$A_{cl} = \begin{bmatrix} \frac{9r^2 - 4r - 4}{r(3r+2)} & -\frac{6r}{3r+2} \\ 7 & -3 \end{bmatrix}$$

and for $r \to 0$

$$A_{cl} \approx \begin{bmatrix} -2/r & 0 \\ 7 & -3 \end{bmatrix}.$$

This shows that one eigenvalue goes to -3 and the other to $-\infty$.

Question 4

(a) Let

$$H = \frac{1}{2}u^2 + \lambda(x^3 + u).$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x} = x^3 + u \qquad \dot{\lambda} = -3\lambda x^2 \qquad 0 = \frac{\partial H}{\partial u} = u + \lambda \qquad 0 = H.$$

(b) The optimal control as a function of λ is $u = -\lambda$. Replacing this into $H = 0$ yields

$$\frac{1}{2}\lambda^2 + \lambda x^3 - \lambda^2 = 0,$$

hence, either

$$\lambda = 0 \qquad u = 0$$

or

$$\lambda = 2x^3 \qquad u = -2x^3.$$

Note that the solution $u = 0$ is not admissible, in fact, if $u = 0$ then $\dot{x} = x^3$, and if $x(0) = x_0 \neq 0$ the condition $x(T) = 0$ cannot be met for any T. The only admissible solution is therefore $u = -2x^3$.

(c) The optimal closed-loop system is

$$\dot{x} = -x^3,$$

and integrating this differential equation with $x(0) = 0$ yields

$$x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}}.$$

We conclude that the condition $x(T) = 0$ is met for $T = +\infty$. The optimal control, as a function of t and x_0, is

$$u(t) = -\frac{2x_0^3}{(\sqrt{2x_0^2 t + 1})^3},$$

and

$$J = \int_0^\infty \frac{1}{2}u^2(t)dt = \frac{1}{2}x_0^4.$$

Question 5

(a) Controllability of (A, B) is equivalent to

$$\text{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n$$

for all $s \in \mathbb{C}$. However,

$$\text{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = \text{rank}\begin{bmatrix} sI - A' \\ B' \end{bmatrix} = \text{rank}\begin{bmatrix} sI + A \\ C \end{bmatrix},$$

hence (A, C) is observable if and only if (A, B) is controllable.

(b)

$$0 = A'P + PA - PBB'P + \alpha C'C = -AP + PA - PBB'P + \alpha BB'.$$

(c) Let $P = \lambda I$, with $\lambda > 0$. Then the ARE becomes

$$0 = -\lambda A + \lambda A - \lambda^2 BB' + \alpha BB',$$

hence $P = \sqrt{\alpha} I$ is a solution of the ARE and it is positive definite. The optimal state feedback control law is

$$u = -Kx = -R^{-1}B'Px = -\sqrt{\alpha}B'x$$

and the optimal closed-loop system is

$$\dot{x} = A_{cl}x = (A - \sqrt{\alpha}BB')x.$$

(d) For the specified A and B, the optimal closed-loop system is

$$\dot{x} = A_{cl}x = \begin{bmatrix} -\sqrt{\alpha} & -1 - \sqrt{\alpha} \\ 1 - \sqrt{\alpha} & -\sqrt{\alpha} \end{bmatrix} x$$

and the characteristic polynomial of the matrix A_{cl} is

$$s^2 + 2s\sqrt{\alpha} + 1.$$

The roots of the characteristic polynomial are

$$-\sqrt{\alpha} + \sqrt{\alpha - 1} \qquad -\sqrt{\alpha} - \sqrt{\alpha - 1}.$$

As $\alpha \to +\infty$ the first tends to 0 and the second to $-\infty$. Finally, the transfer function of the system is

$$W(s) = C(sI - A)^{-1}B = \frac{2s}{s^2 + 1}.$$

This has a zero for $s = 0$, and this is where one of the eigenvalues of the optimal closed-loop system tends as $\alpha \to +\infty$.

Question 6

(a) Let

$$H = 1 + \lambda_1 x_2 + \lambda_2 u.$$

The necessary conditions of optimality, for normal extremals, are

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u \qquad \dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = -\lambda_1 \qquad 0 = 1 + \lambda_1 x_2 + \lambda_2 u$$

$$1 + \lambda_1 x_2 + \lambda_2 u \leq 1 + \lambda_1 x_2 + \lambda_2 \omega, \ \forall \omega \in [-1, 1].$$

(b) From the last condition the optimal control is

$$u = -\text{sign}(\lambda_2),$$

hence $u = \pm 1$.

(c) If $u = c$, for some constant c, then

$$x_1(t) = x_1(0) + x_2(0)t + ct^2/2 \qquad\qquad x_2(t) = x_2(0) + ct.$$

If $u = +1$, $x_1(0) = -1/2$ and $x_2(0) = 1$, then

$$x_1(t) = -1/2 + t + t^2/2 \qquad\qquad x_2(t) = 1 + t$$

and the condition $x_2(t_f) = 0$ cannot be met for any t_f. If $u = -1$, $x_1(0) = -1/2$ and $x_2(0) = 1$

$$x_1(t) = -1/2 + t - t^2/2 \qquad\qquad x_2(t) = 1 - t$$

and the condition $x_2(t_f) = x_1(t_f) = 0$ holds with $t_f = 1$.

(d) Integration of the costate equations yields

$$\lambda_1(t) = \lambda_1(0) \qquad \lambda_2(t) = \lambda_2(0) - \lambda_1(0)t.$$

Substituting into the Hamiltonian yields

$$H = 1 + \lambda_1(0)(1 - t) - \lambda_2(0) + \lambda_1(0)t = 1 + \lambda_1(0) - \lambda_2(0).$$

Hence, $H = 0$ for all t (along the optimal solutions) if

$$1 + \lambda_1(0) - \lambda_2(0) = 0.$$

(e) Let $z = -x$ and note that

$$\dot{z}_1 = z_2 \qquad\qquad \dot{z}_2 = -u$$

and $z(0) = -x(0)$. So the optimization problem in the z variables is solved by $-u = -1$ for $t \in [0, 1]$.