IMPERIAL COLLEGE LONDON

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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

MSc and EEE/ISE PART IV: MEng and ACGI

DIGITAL SIGNAL PROCESSING AND DIGITAL FILTERS

Tuesday, 16 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Corrected Copy

Q1

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : A.G. Constantinides
Second Marker(s) : M.K. Gurcan

figure is correct.   + 5 min time.

The Questions

1. The signal flow graph of an A/D converter based on oversampling techniques is shown in Figure 1. The block indicated by S in the figure is a two-input, single-output linear system described by $U = \alpha X + \beta V$ where α and β are appropriate transfer functions which may be frequency dependent. The block labelled $Q[\bullet]$ is a bipolar one-bit quantiser, which introduces quantisation noise Q as indicated.

10:05

i) By assuming the quantiser to be linear and by making additional appropriate assumptions derive an expression for the output Y in terms of X, α and β and the quantisation noise Q. Comment on the validity of your assumptions in practice. [6]

In a specific realisation it is required that the following conditions be satisfied: a) the output needs to have real unity gain with respect to the input, and b) the noise shaping transfer function is required to be $F(z)$.

ii) Show that under these conditions $\alpha = \dfrac{1}{F(z)}$ and $\beta = \dfrac{F(z)-1}{F(z)}$. [4]

iii) Give an account of the factors that influence the choice for $F(z)$. [5]

iv) Draw the signal flow graph of the interconnecting block **S** when $F(z) = (1 - z^{-1})$ and reduce it to a form that contains only one accumulator. [5]

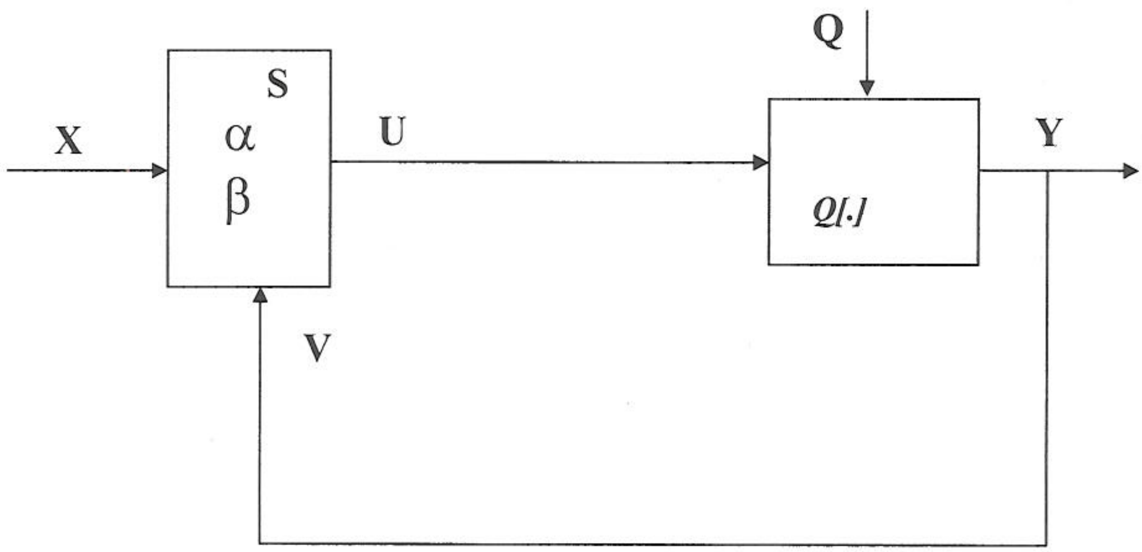


Figure 1

2. i) Show that the real transfer function

$$A_1(z,\alpha) = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}} \qquad |\alpha| < 1$$

is allpass. [2]

Hence show that for real β the transfer function

$$A_2(z,\alpha,\beta) = A_1\left(-z^{-1}A_1(z,\alpha),\beta\right)$$

is also allpass. [3]

ii) Determine the non-trivial frequency θ_0 at which $A_2(z,\alpha,\beta)$ is completely real and hence provide arguments to support the view that $H_1(z)$ is bandpass and $H_2(z)$ is a complementary bandstop, where

$$H_1(z) = \frac{1}{2}\left(1 - A_2(z,\alpha,\beta)\right)$$

$$H_2(z) = \frac{1}{2}\left(1 + A_2(z,\alpha,\beta)\right)$$

Hence determine the centre frequencies of the two filters. [6]

iii) Show that the bandpass filter attains its 3dB values at the frequencies that satisfy the condition

$$\mathrm{Re}\left[A_2(z,\alpha,\beta)\right] = 0 \qquad \text{for } |z| = 1$$

[6]

iv) Comment on the practical utility of these results. [3]

4. An infinite impulse response digital filter transfer function $H(z)$ is to be realised by the structure shown in Figure 2 as $H(z) = Y_1 / X_1$ The constraining transfer function is $z^{-1}C(z)$ and the relationship between $H(z)$ and $z^{-1}C(z)$ is given by

$$H(z) = \frac{\alpha + z^{-1}C(z)}{1 + \alpha z^{-1}C(z)}$$

where $-1 < \alpha < 1$.

i) Determine a set of {a b c d} parameters for the interconnecting system. [7]

ii) Comment on the choice available and its implications. Suggest a good selection and justify your answers [5]

iii) If $|C(z)| < 1$ on $|z| = 1$ show that $|H(z)| < 1$. [8]

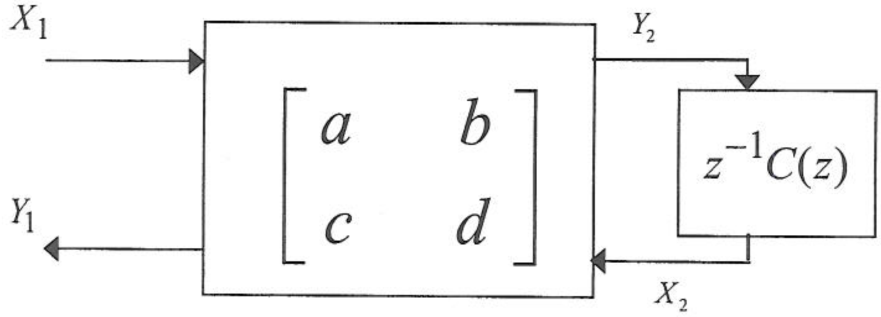


Figure 2

5. i) Define the root moments $\{S_m\}$ of a Finite Impulse Response transfer function

$$H(z) = K\prod_{i=1}^{n_1}\left(1-\alpha_i z^{-1}\right)\prod_{i=1}^{n_2}\left(1-\beta_i z^{-1}\right)$$

where m is the degree of the moment, K is a constant, α_i are the zeros inside the unit circle and β_i are the zeros outside the unit circle and show that if $H(z)$ is real then so are its root moments [2]

ii) If $H(z)$ is minimum phase show that its root moments decrease exponentially with the index m. [2]

iii) Show that if the amplitude and phase responses are respectively $A(\theta)$ and $\phi(\theta)$ then

$$\ln\left(A(\theta)\right) = \ln\left(K_1\right) - \sum_{m=1}^{\infty}\frac{S_m^{N_1} + S_{-m}^{N_2}}{m}\cos(m\theta)$$

$$\phi(\theta) = -n_2\theta + \sum_{m=1}^{\infty}\frac{S_m^{N_1} - S_{-m}^{N_2}}{m}\sin(m\theta)$$

where K_1 is an appropriate real constant, $S_m^{N_1}$ are the root moments of the minimum phase factor and $S_{-m}^{N_2}$ the inverse root moments of the maximum phase factor of $H(z)$. [12]

iv) Hence, show that for the transfer function $H(z)$ to have linear phase it must have zeros located outside the unit circle, and determine their number and location in relation to the number of zeros located inside the unit circle. [4]

Q1.

<u>Assumptions:</u>

1) Sampling rate is high enough to enable X to be represented by its z-transform equivalent
2) Quantisation model is linear.

ie

3) Loop is computable ie it contains at least one sample delay
4) Loop is stable ie no poles outside $|z|=1$
5) Computational latencies are negligible ie multiplication & quantisation take insignificant time w.r.t. sampling period.

<u>Comments on above:</u>

1) Realisable within a reasonable upper limit
2) Oversimplification - can lead to results unpredictable from linear analysis
3) Easily achievable
4) Can be made to be so
5) Can lead to problems (stability) if latencies are comparable to sampling periods.

From the figure we can write directly (in the transform domain)

$$\alpha X + \beta Y + Q = Y$$

or $$Y = \frac{\alpha}{1-\beta} X + \frac{1}{1-\beta} Q$$

The linearity oversimplification may invalidate

2/10

the analysis.
Requirement (a) imposes ~~the~~ condition 5/6

$$\alpha = 1-\beta .$$

[The condition $\alpha = -(1-\beta)$ is equally acceptable if only the magnitude is required]
Requirement (b) produces

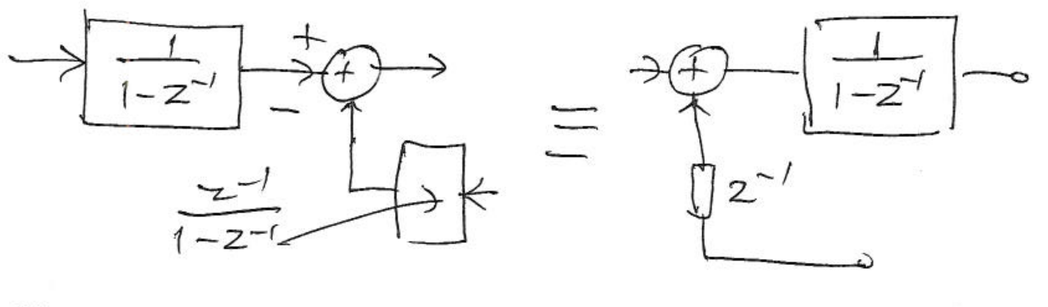
$$\frac{1}{1-\beta} = F(z) = \frac{1}{\alpha}$$

Hence $\beta = 1 - \frac{1}{F(z)} = (F(z)-1)/F(z)$ 4/4

The noise shaping filter $F(z)$ is selected such that the quantisation noise spectrum at the output is low within the signal bandwidth.
The noise power outside the band is removed by post filtering.
~~If~~ x is lowpass then $F(z)$ is highpass
For

$F(z) = 1 - z^{-1}$ we have
$\alpha = 1/(1-z^{-1})$ and $\beta = -z^{-1}/(1-z^{-1})$ 5/5

The connecting block is then



Two accumulators   Single accumulator 5/6

20

$\frac{7}{10}$

Q2

Let $A(z,\alpha) = \dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}$ as $z^{-1} \cdot \dfrac{1+\alpha z}{1+\alpha z^{-1}}$

For $z = e^{j\theta}$ & α real we have

$|z| = 1$ $\quad (1+\alpha e^{j\theta}) = (1+\alpha e^{-j\theta})^*$

Hence $\quad |A(z,\alpha)|_{\text{on } |z|=1} = 1$ $\quad$ 3/3

Let $z^{-1}A_1(z,\alpha) = e^{j\phi(\theta)}$ on $z = e^{j\theta}$

then

$$A_2(z,\alpha,\beta)\Big|_{e^{j\theta}} = A_1(e^{-j\theta}\cdot A_1(e^{j\theta},\alpha),\beta)$$

$$= \frac{e^{j\phi(\theta)}+\beta}{1+\beta e^{j\phi(\theta)}}$$

3/3

and by similar arguments $|A_2(z,\alpha,\beta)| = 1$

Obtain $A_2(z,\alpha,\beta)$ as

$$A_2(z,\alpha,\beta) = \frac{z^{-1}\cdot\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}+\beta}{1+\beta z^{-1}\cdot\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}} = \frac{z^{-2}+\alpha(1+\beta)z^{-1}+\beta}{\beta z^{-2}+\alpha(1+\beta)z^{-1}+1}$$

Since $|A_2|_{e^{j\theta}} = 1$ the only real values it can have are $+1$ and -1.

For

$A_2 = +1$ $\quad \beta z^{-2}+\alpha(1+\beta)z^{-1}+1 = z^{-2}+\alpha(\beta+1)z^{-1}+\beta$

or $\quad (\beta-1)z^{-2} = \beta - 1$ $\quad$ ie $\quad z = \pm 1$

These are the trivial solutions

For

$A_2 = -1$ $\quad -\beta z^{-2} - \alpha(1+\beta)z^{-1} - 1 = z^{-2} + \alpha(1+\beta)z^{-1} + \beta$

or

$$(1+\beta)z^{-2} + 2\alpha(1+\beta)z^{-1} + (1+\beta) = 0$$

$\beta \neq -1$

$$z^{-2} + 2\alpha z^{-1} + 1 = 0$$

Since $|\alpha| < 1$ $\quad$ let $\alpha = -\cos\mu$

$$z^{-2} - 2\cos\mu\, z^{-1} + 1 = (z^{-1} - e^{-j\mu})(z^{-1} - e^{+j\mu})$$

i.e. $\theta_0 = \cos^{-1}\alpha$.

At θ_0 $\quad H_1(e^{j\theta_0}) = 1 \quad H_2(e^{j\theta_0}) = 0$ 6/6

Since the equations are quadratic we expect the phase angle of A_2 to be such that $e^{j\phi(\theta)}$ assumes the same value twice, once below and again above θ_0. Hence the responses will be bandpass and bandstop respectively. The centre frequencies are both at θ_0.

For the bandpass case the 3dB points are at those frequencies at which

$$\left(\tfrac{1}{2}\right)^2 |1 - A_2|^2 = \tfrac{1}{2} \quad \text{or} \quad |1 - A_2|^2 = 2$$

But $|1 - A_2|^2 = |1 - A_2||1 - A_2^*| = |1 + |A_2|^2 - (A_2 + A_2^*)|$

$$= |2 - (A_2 + A_2^*)|$$

Since $|A_2| = 1 = e^{j\phi(\theta)}$ $\quad A_2 + A_2^* = 2\cos\phi$

or $|A_2 + A_2^*| < 2$

i.e.

$$2 - (A_2 + A_2^*) = 2 \quad \text{or} \quad A_2 + A_2^* = 0$$

Thus the 3dB points correspond to the frequencies satisfying 6/6

$$\mathrm{Re}\{A_2(z, \alpha, \beta)\}_c = 0$$

The arrangement can be used for bandsplitting applications that underpin areas such as compression. 3/3

Q3

(i) Let $A_1(e^{j2\theta}) = e^{j\phi_1(\theta)}$, $A_2(e^{j2\theta}) = e^{j(\phi_2(\theta)+\theta)}$

Then

$$2\,G(e^{j\theta}) = e^{j\phi_1(\theta)} + e^{-j\theta} \cdot e^{+j\theta} \cdot e^{j\phi_2(\theta)} = e^{j\phi_1(\theta)} + e^{j\phi_2(\theta)}$$

or

$$G(e^{j\theta}) = e^{j\left(\frac{\phi_1(\theta)+\phi_2(\theta)}{2}\right)} \left[\frac{e^{j\frac{(\phi_1(\theta)-\phi_2(\theta))}{2}} + e^{-j\frac{(\phi_1(\theta)-\phi_2(\theta))}{2}}}{2}\right]$$

Therefore $|G(e^{j\theta})| = \cos\frac{\Delta\phi(\theta)}{2}$

where

$$\Delta\phi(\theta) = \phi_1(\theta) - \phi_2(\theta)$$

(ii) In the passband the minimum value occurs at several points including $\theta = \theta_c$

ie

$$1 - \epsilon_1 = \left|\cos\frac{\Delta\phi(\theta_c)}{2}\right|$$

$$\text{or } \epsilon_1 = 1 - \left|\cos\frac{\Delta\phi(\theta_c)}{2}\right|$$

In the stopband, similarly, the maximum value occurs at $\theta = \theta_1$

hence

$$\epsilon_2 = \left|\cos\frac{\Delta\phi(\theta_1)}{2}\right|$$

iii) When θ is replaced by $\theta+\pi$ we have a rotation of $180°$ ie z is replaced by $-z$ and the P.B. and SB are interchanged. ie a LP response becomes a HP response. Since both $A_1(z^2)$ and $A_2(z^2)$ are functions of z^2 they remain unchanged & hence the new transfer function is

$$G_1(z) = (A_1(z^2) - z^{-1}A_2(z^2))/2$$

which is highpass

5/10

4/4

6/6

6/6

6/10

iv) The replacement $\theta \to \pi+\theta$ makes the LP cutoff frequency $-\theta_c$ (negative) go to the positive HP cutoff frequency $\pi-\theta_c$.

If the impulse response of the LP is $h(n)$ then

$$G(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

and hence

$$G_1(z) = \sum_{n=0}^{\infty} h(n) \cdot (-z)^{-n} = \sum_{n=0}^{\infty} h_{HP}(n) \cdot z^{-n}$$

or

$$h_{HP}(n) = (-1)^n \cdot h(n) .$$

4/4

7/10

Q4

Write $Y_1 = a X_1 + b X_2$

$Y_2 = c X_1 + d X_2$

and $X_2 = z^{-1} C\, Y_2$

Hence

$Y_2 = c X_1 + d z^{-1} C Y_2$

$$\text{or } Y_2 = \frac{c}{1 - d z^{-1} c} \cdot X_1$$

Thus

$$Y_1 = a X_1 + b z^{-1} C \cdot \frac{c}{1 - d z^{-1} C} \cdot X_1$$

or

$$\frac{Y_1}{X_1} = \frac{a + (bc - ad) \cdot z^{-1} C(z)}{1 - d z^{-1} C(z)}$$

Compare given expression with above

$d = -\alpha$

$a = \alpha$

$bc - ad = 1 \rightarrow bc = 1 + da = 1 - \alpha^2$ 7/7

There are 3 equations with 4 parameters

For <u>minimal multiplier</u> solution

either $b = 1$ & $c = 1 - \alpha^2$ or v.v.

Now consider 5/5

$$1 - |H(z)|^2 = 1 - \left| \frac{\alpha + z^{-1} C(z)}{1 + \alpha z^{-1} C(z)} \right|^2$$

with α real

$$\text{or } 1 - |H(z)|^2 = 1 - \frac{[\alpha + z^{-1} C(z)]\,[\alpha + z\, C^*(z)]}{[1 + \alpha z^{-1} C(z)]\,[1 + \alpha z\, C^*(z)]}$$

8/10

or

$$1-|H(z)|^2 = 1 - \frac{[\alpha^2 + \alpha(z^{-1}C(z) + zC^*(z)) + |C(z)|^2]}{[1 + \alpha(z^{-1}C(z) + zC^*(z)) + \alpha^2 |C(z)|^2}$$

$$= \frac{1 + \alpha^2 |C(z)|^2 - \alpha^2 - |C(z)|^2}{\text{Squared Real Quantity}}$$

$$= \frac{(1-\alpha^2)[1-|C(z)|^2]}{\text{Positive}}$$

i.e. $1 - |H(z)|^2 \geq 0$ when $|C(z)|^2 \leq 1$ on $|z|=1$

8/8

9/10

Q5

With $H(z) = K \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$

The root moments are

$$S_m = \sum_{i=1}^{n_1} \alpha_i^m + \sum_{i=1}^{n_2} \beta_i^m \qquad m \text{ integer} > 0$$ 2/2

So then if $H(z)$ is real, α_i & β_i are in conjugate form and hence S_m are real

a) If $H(z)$ is minimum phase then $\beta_i = 0$ and $|\alpha_i| < 1 \quad \forall i$; hence the root moments decrease exponentially with m. 2/2

b) On $z = e^{j\theta}$ let $H(z)\big|_{e^{j\theta}} = A(\theta) e^{j\phi(\theta)}$

Moreover use $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ so that

$$\ln A(\theta) + j\phi(\theta) = \ln K + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln\left\{(-\beta_i z^{-1})\left(1 - \frac{z}{\beta_i}\right)\right\}$$

$$= \ln K_1 - n_2 \ln z + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln\left(1 - \frac{z}{\beta_i}\right)$$

with $z = e^{j\theta}$, $\ln K_1 = \ln\{K \prod(-\beta_i)\}$

$$\ln A(\theta) + j\phi(\theta) = \ln K_1 - j n_2 \theta - \sum_{m=1}^{\infty} \frac{S_m}{m} \cdot z^{-1} + \frac{S_{-m}}{m} \cdot z \quad \Big|_{z = e^{j\theta}}$$

where $S_m^{N_1} = \sum_{i=1}^{n_1} \alpha_i^m \qquad S_{-m}^{N_2} = \sum_{i=1}^{n_2} \beta_i^{-m}$

Therefore,

$$\ln A(\theta) = \ln K_1 - \sum_{m=1}^{\infty} \frac{S_m^{N_1} - S_{-m}^{N_2}}{m} \cdot \cos m\theta$$

$$\phi(\theta) = -n_2 + \sum_{m=1}^{\infty} \frac{S_m^{N_1} - S_{-m}^{N_2}}{m} \cdot \sin m\theta$$ 12/12

10/10

c) If $\phi(\theta)$ is linear in θ then

$$S_m^{N_1} \equiv S_{-m}^{N_2} \qquad \forall m = 1, 2, \ldots$$

i.e. there must be zeros located outside the unit circle the inverse moments of which are identically equal to those within the circle.

This can only happen when there are as many zeros outside as there are inside the circle and located reciprocally to one another. 4/4