



1. (a) An  $M^{\text{th}}$ -order, causal, real transfer function is given as

$$A_M(z) = \frac{a_0 + a_1 z^{-1} + \dots + a_M z^{-M}}{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}, \quad M > 0$$

(i) Derive the conditions on the numerator and denominator coefficients to yield  $|A_M(e^{j\omega})|^2 = 1$ , for all  $\omega$ .

[3]

(ii) Comment on the location of the zeros of a causal, real, stable, allpass transfer function within the  $z$ -plane with respect to the unit circle and the poles. Justify your answer.

[3]

(b) Consider the first-order, causal, real, stable, allpass transfer function  $A_1(z)$ .

(i) Determine the expression for  $1 - |A_1(z)|^2$ .

[5]

(ii) Show that

$$|A_1(z)| \begin{cases} < 1 & \text{for } |z| > 1 \\ = 1 & \text{for } |z| = 1 \\ > 1 & \text{for } |z| < 1 \end{cases}$$

[5]

(c) Discuss a possible application of allpass filters.

[4]

2. (a) Show that the transfer function

$$H_1(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (1)$$

has a lowpass magnitude response. Determine the 3-dB cut-off frequency  $\omega_c$  at which the gain response is 3 dB below the maximum value of 0 dB at  $\omega = 0$ .

[5]

- (b) Show that the transfer function

$$H_2(z) = \frac{1+\alpha}{2} \frac{1-z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (2)$$

has a highpass magnitude response. Determine the 3-dB cut-off frequency  $\omega_c$  at which the gain response is 3 dB below the maximum value of 0 dB at  $\omega = \pi$ .

[5]

- (c) The lowpass transfer function  $H_1(z)$  of equation (1) and the highpass transfer function  $H_2(z)$  of equation (2) can be expressed in the form

$$H_1(z) = \frac{1}{2} [A_1(z) - A_2(z)], \quad H_2(z) = \frac{1}{2} [A_1(z) + A_2(z)]$$

where  $A_1(z)$  and  $A_2(z)$  are stable, allpass transfer functions. Determine  $A_1(z)$  and  $A_2(z)$ .

[10]

3. (a) Show that a Type 1 linear-phase FIR transfer function  $H(z)$  of length  $2M + 1 = 7$  can be expressed as

$$H(z) = z^{-M} \left[ h[M] + \sum_{n=1}^M h[M-n](z^n + z^{-n}) \right] \quad [4]$$

- (b) By using the relation

$$z^r + z^{-r} = 2T_r\left(\frac{z + z^{-1}}{2}\right)$$

where  $T_r(x)$  is the  $r$ -th order Chebyshev polynomial in  $x$ , express  $H(z)$  of Part (a) in the form

$$H(z) = z^{-M} \sum_{n=0}^M a[n] \left( \frac{z + z^{-1}}{2} \right)^n \quad (1)$$

Determine the relation between  $a[n]$  and  $h[n]$ . [8]

- (c) Develop a realization of  $H(z)$  based on equation (1) in the form of Figure 1, where  $F_1(z^{-1})$  and  $F_2(z^{-1})$  are causal structures. Determine the form of  $F_1(z^{-1})$  and  $F_2(z^{-1})$ . [8]

[The Chebyshev polynomials satisfy the following recursive relationship:

$$T_r(x) = 2xT_{r-1}(x) - T_{r-2}(x), r \geq 2$$

$$T_0(x) = 1, T_1(x) = x]$$

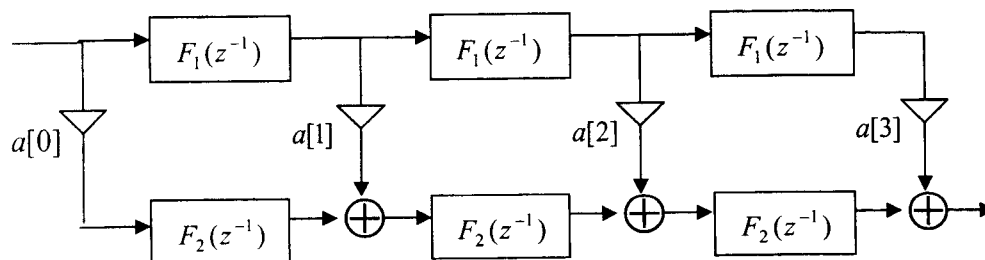


Figure 1

4. (a) The bilinear transformation from the  $s$  – plane to the  $z$  – plane is given by

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

(i) Explain using mathematical relations, where each point  $s = \sigma + j\Omega$  on the  $s$  – plane is mapped on the  $z$  – plane through the bilinear transformation. [6]

(ii) Prove that the relation between the continuous frequency  $\Omega$  and the discrete frequency  $\omega$  is non-linear. [4]

(b) A given real lowpass digital IIR filter has a rational transfer function  $H_L(z)$  and a digital cutoff frequency  $\omega_c$ . The transfer function is transformed by replacing  $z$  by  $F(\hat{z})$ , to another real lowpass rational form  $H_D(\hat{z}) = H_L(F(\hat{z}))$  of the same order as  $H_L(z)$  and cutoff frequency  $\hat{\omega}_c$ .

(i) Give a full account of the properties of  $F(\hat{z})$  in order to effect the transformation. [3]

(ii) Starting with a general expression for  $F(\hat{z})$  find a specific expression for  $F(\hat{z})$  which has all its parameters derivable from the requirements of the problem. [7]

- 5 (a) (i) Explain the function of a down sampler with a down sampling factor  $M$  where  $M$  is a positive integer. Give the relation between the output sequence  $y[n]$  of a down sampler with a down sampling factor  $M$  and its input sequence  $x[n]$ . [4]
- (ii) Derive the relation between the spectrums of the input and the output of a down sampler. [8]
- (b) Determine the condition under which a cascade of a factor of  $M$  down sampler with a factor of  $L$  up sampler is interchangeable. [8]

1. (a) (i) An  $M^{\text{th}}$ -order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \dots + d_1z^{-M+1} + z^{-M}}{1 + d_1z^{-1} + \dots + d_{M-1}z^{-M+1} + d_Mz^{-M}}$$

If we denote the denominator polynomial of the allpass function  $A_M(z)$  as  $D_M(z)$  with

$$D_M(z) = 1 + d_1z^{-1} + \dots + d_{M-1}z^{-M+1} + d_Mz^{-M}$$

then it follows that  $A_M(z)$  can be written as

$$A_M(z) = \pm \frac{z^{-M}D_M(z^{-1})}{D_M(z)}$$

From the above we see that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M}D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})} = 1 \Rightarrow$$

$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1}) = 1$$

(ii) The poles of a causal stable transfer function must lie inside the unit circle. As a result, all zeros of a causal stable allpass transfer function lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle.

(b) Consider the first-order causal and stable allpass transfer function  $A_1(z)$  with its pole located at  $d_1$ .

(i) Determine the expression for  $1 - |A_1(z)|^2$ .

$$A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1z^{-1}} = \frac{zd_1 + 1}{z + d_1} \text{ with } d_1 \text{ real and } |d_1| < 1.$$

$$A_1^*(z) = \frac{z^*d_1 + 1}{z^* + d_1}$$

$$A_1(z)A_1^*(z) = \frac{zd_1 + 1}{z + d_1} \cdot \frac{z^*d_1 + 1}{z^* + d_1} = \frac{|z|^2 d_1^2 + d_1(z + z^*) + 1}{|z|^2 + d_1(z + z^*) + d_1^2}$$


$$1 - A_1(z)A_1^*(z) = 1 - \frac{|z|^2 d_1^2 + d_1(z + z^*) + 1}{|z|^2 + d_1(z + z^*) + d_1^2} = \frac{(|z|^2 - 1)(1 - d_1^2)}{|z + d_1|^2}$$

(ii) If  $|z| > 1$  then  $|z|^2 - 1 > 0$ . Since  $\frac{(1 - d_1^2)}{|z + d_1|^2} > 0$  we see that

$$1 - A_1(z)A_1^*(z) = \frac{(|z|^2 - 1)(1 - d_1^2)}{|z + d_1|^2} > 0 \Rightarrow |A_1(z)|^2 < 1.$$

The rest are proved in a similar fashion.

(c) A simple but often used application of an allpass filter is as a **delay equalizer**. Let  $G(z)$  be the transfer function of a digital filter that has been designed to meet a prescribed

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magnitude response. The nonlinear phase response of this filter can be corrected by cascading it with an allpass filter section  $A(z)$  so that the overall cascade with transfer function  $G(z)A(z)$  as a constant group delay over the frequency domain of interest. Since the allpass filter has a unity magnitude response, the magnitude response of the cascade is still equal to  $|G(e^{j\omega})|$ , while the overall delay is given by the sum of the group delays of  $G(z)$  and  $A(z)$ . The allpass is designed so that the overall group delay is approximately a constant in the frequency region of interest.

2 (a) For the transfer function

$$H_1(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (1)$$

we have  $H_1(j\omega) = \frac{1-\alpha}{2} \frac{1+e^{-j\omega}}{1-\alpha e^{-j\omega}}, |\alpha| < 1,$

$$|H_1(j\omega)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos\omega)^2 + \sin^2\omega}{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega} = \frac{(1-\alpha)^2}{2} \frac{1+\cos\omega}{1+\alpha^2-2\alpha\cos\omega}.$$

The above has a lowpass magnitude response with a monotonically decreasing magnitude since

$$|H_1(j0)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos 0)^2 + \sin^2 0}{(1-\alpha\cos 0)^2 + \alpha^2\sin^2 0} = \left(\frac{1-\alpha}{2}\right)^2 \frac{4}{(1-\alpha)^2} = 1,$$

$$|H_1(j\pi)|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1+\cos\pi)^2 + \sin^2\pi}{(1-\alpha\cos\pi)^2 + \alpha^2\sin^2\pi} = \left(\frac{1-\alpha}{2}\right)^2 \frac{(1-1)^2 + 0}{(1+\alpha)^2 + 0} = 0$$

$$\begin{aligned} \frac{d(|H_1(j\omega)|^2)}{d\omega} &= \frac{(1-\alpha)^2}{2} \frac{(-\sin\omega)(1+\alpha^2-2\alpha\cos\omega) - (1+\cos\omega)2\alpha\sin\omega}{(1+\alpha^2-2\alpha\cos\omega)^2} \\ &= \frac{(1-\alpha)^2}{2} \frac{\sin\omega(-1-\alpha^2+2\alpha\cos\omega-2\alpha-2\alpha\cos\omega)}{(1+\alpha^2-2\alpha\cos\omega)^2} = -\frac{(1-\alpha)^2}{2} \frac{(1+\alpha)^2\sin\omega}{(1+\alpha^2-2\alpha\cos\omega)^2} \end{aligned}$$

which is monotonically decreasing within the range  $\omega = [0, \pi]$ .

The 3-dB cut-off frequency  $\omega_c$  at which the gain response is 3 dB below the maximum value of 0 dB at  $\omega = 0$  is found below

$$|H_1(j\omega_c)|^2 = \frac{(1-\alpha)^2}{2} \frac{1+\cos\omega_c}{1+\alpha^2-2\alpha\cos\omega_c} = \frac{1}{2} \Rightarrow \cos\omega_c = \frac{2\alpha}{1+\alpha^2}$$

(b) For the transfer function

$$H_2(z) = \frac{1+\alpha}{2} \frac{1-z^{-1}}{1-\alpha z^{-1}}, |\alpha| < 1 \quad (2)$$

we have

$$|H_2(j\omega)|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-\alpha\cos\omega)^2 + \alpha^2\sin^2\omega} = \frac{(1+\alpha)^2}{2} \frac{1-\cos\omega}{1+\alpha^2-2\alpha\cos\omega}.$$

The above has a highpass magnitude response with a monotonically increasing magnitude since



$$\begin{aligned}
|H_2(j0)|^2 &= \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos 0)^2 + \sin^2 0}{(1-\alpha \cos 0)^2 + \alpha^2 \sin^2 0} = 0, \\
|H_2(j\pi)|^2 &= \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos \pi)^2 + \sin^2 \pi}{(1-\alpha \cos \pi)^2 + \alpha^2 \sin^2 \pi} = \left(\frac{1+\alpha}{2}\right)^2 \frac{(1+1)^2}{(1+\alpha)^2} = 1 \\
|H_2(j\omega)|^2 &= \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-\cos \omega)^2 + \sin^2 \omega}{(1-\alpha \cos \omega)^2 + \alpha^2 \sin^2 \omega} = \frac{(1+\alpha)^2}{2} \frac{1-\cos \omega}{1+\alpha^2-2\alpha \cos \omega} \\
\frac{d(|H_2(j\omega)|^2)}{d\omega} &= \frac{(1+\alpha)^2}{2} \frac{\sin \omega(1+\alpha^2-2\alpha \cos \omega) - (1-\cos \omega)2\alpha \sin \omega}{(1+\alpha^2-2\alpha \cos \omega)^2} \\
&= \frac{(1+\alpha)^2}{2} \frac{\sin \omega(1+\alpha^2-2\alpha \cos \omega - 2\alpha + 2\alpha \cos \omega)}{(1+\alpha^2-2\alpha \cos \omega)^2} = \frac{(1+\alpha)^2}{2} \frac{(1-\alpha)^2 \sin \omega}{(1+\alpha^2-2\alpha \cos \omega)^2}
\end{aligned}$$

which is monotonically increasing within the range  $\omega = [0, \pi]$ .

The 3-dB cut-off frequency  $\omega_c$  at which the gain response is 3 dB below the maximum value of 0 dB at  $\omega = 0$  is found below

$$|H_1(j\omega_c)|^2 = \frac{(1+\alpha)^2}{2} \frac{1-\cos \omega_c}{1+\alpha^2-2\alpha \cos \omega_c} = \frac{1}{2} \Rightarrow \cos \omega_c = \frac{2\alpha}{1+\alpha^2}$$

(c)  $A_1(z) = 1 \quad A_2(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}} \quad |A_1(j\omega)| = 1 \quad |A_2(j\omega)| = 1$

3. (a)

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$$

$$h[6] = h[0], h[5] = h[1], h[4] = h[2]$$

$$H(z) = h[0](1 + z^{-6}) + h[1](z^{-1} + z^{-5}) + h[2](z^{-2} + z^{-4}) + h[3]z^{-3} \Rightarrow$$

$$H(z) = z^{-3} \{h[3] + h[0](z^3 + z^{-3}) + h[1](z^2 + z^{-2}) + h[2](z + z^{-1})\}$$

Thus

$$H(z) = z^{-3} \left[ h[3] + \sum_{n=1}^3 h[3-n](z^n + z^{-n}) \right]$$

(b) The Chebychev polynomials satisfy the following recursive relationship:

$$z^r + z^{-r} = 2T_r\left(\frac{z + z^{-1}}{2}\right)$$

with  $T_r(x) = 2xT_{r-1}(x) - T_{r-2}(x)$ ,  $r \geq 2$   $T_0(x) = 1$   $T_1(x) = x$ .

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$$

Thus,

$$z^2 + z^{-2} = 2T_2\left(\frac{z + z^{-1}}{2}\right) = 4\left(\frac{z + z^{-1}}{2}\right)^2 - 2$$

$$z^3 + z^{-3} = 2T_3\left(\frac{z+z^{-1}}{2}\right) = 8\left(\frac{z+z^{-1}}{2}\right)^3 - 6\left(\frac{z+z^{-1}}{2}\right)$$

For reasons of simplicity I call  $\frac{z+z^{-1}}{2} = x$

$$\begin{aligned} H(z(x)) &= z^{-3}(h[3]x^0 + h[0](8x^3 - 6x) + h[1](4x^2 - 2) + 2h[2]x) \\ &= z^{-3}[(h[3] - 2h[1])x^0 + (2h[2] - 6h[0])x + 4h[1]x^2 + 8h[0]x^3] \end{aligned}$$

$$H(z) = z^{-M} \sum_{n=0}^M a[n] \left(\frac{z+z^{-1}}{2}\right)^n$$

$$a[0] = h[3] - 2h[1], a[1] = 2h[2] - 6h[0], a[2] = 4h[1], a[3] = 8h[0]$$

- (c) Develop a realization of  $H(z)$  based on equation (1) in the form of Figure 1, where  $F_1(z^{-1})$  and  $F_2(z^{-1})$  are causal structures. Determine the form of  $F_1(z^{-1})$  and  $F_2(z^{-1})$ .

$$F_1(z^{-1}) = \frac{z+z^{-1}}{2}, F_2(z^{-1}) = z^{-1}$$

4. (a) (i)  $z = \frac{1+s}{1-s}$

- (ii) For  $s = j\Omega_0$  we have that  $z = \frac{1+j\Omega_0}{1-j\Omega_0}$  which has a unity magnitude. This implies that a point on the imaginary axis in the  $s$ -plane is mapped onto a point on the unit circle in the  $z$ -plane where  $|z|=1$ . In the general case, for  $s = \sigma_0 + j\Omega_0$ ,

$$z = \frac{1+(\sigma_0 + j\Omega_0)}{1-(\sigma_0 + j\Omega_0)} \Rightarrow |z|^2 = \frac{(1+\sigma_0)^2 + \Omega_0^2}{(1-\sigma_0)^2 + \Omega_0^2}$$

A point in the left-half  $s$ -plane with  $\sigma_0 < 0$  is mapped onto a point inside the unit circle in the  $z$ -plane as  $|z| < 1$ . Likewise, a point in the right-half  $s$ -plane with  $\sigma_0 > 0$  is mapped onto a point outside the unit circle in the  $z$ -plane as  $|z| > 1$ .

- (iii) Prove that the relation between the continuous frequency  $\Omega$  and the discrete frequency  $\omega$  is non-linear.

$$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j \tan\left(\frac{\omega}{2}\right)$$

$$\Omega = \tan\left(\frac{\omega}{2}\right)$$

- (b) (i)  $F(\hat{z})$  must be real, rational function  $\hat{z}$ , of order 1 and stable.  
(ii) Starting with a general expression for  $F(\hat{z})$  as follows

$$F(\hat{z}) = \frac{a\hat{z} + b}{c\hat{z} + d}, a, b, c, d \text{ constant parameters}$$

For  $\hat{\omega} = 0$ ,  $\hat{z} = e^{j0} = 1$  we require  $H_D(\hat{z}) = H_L(F(\hat{z})) = 1$ . Thus

$$F(1) = \frac{a+b}{c+d} = 1 \Rightarrow a+b = c+d$$

For  $\hat{\omega} = \pi$ ,  $\hat{z} = e^{j\pi} = -1$  we require  $H_D(\hat{z}) = H_L(F(\hat{z})) = 0$ . Thus

$$F(-1) = \frac{-a+b}{-c+d} = -1 \Rightarrow -a+b = c-d$$

From the above conditions we see that  $b=c$  and  $a=d$ .

$$\text{Therefore, we have } F(\hat{z}) = \frac{a\hat{z}+b}{b\hat{z}+a} = \frac{\alpha\hat{z}+1}{\hat{z}+\alpha}, \text{ with } \alpha = \frac{a}{b}.$$

Therefore,  $F(\hat{z})$  is an allpass filter.

5. (a) (i) The down sampler with a down sampling factor  $M$ , where  $M$  is a positive integer, develops an output sequence  $y[n]$  with a sampling rate that is  $(1/M)^{\text{th}}$  of that of the input sequence  $x[n]$ . The down sampling operation is implemented by keeping every  $M^{\text{th}}$  sample of the input sequence and removing  $M-1$  in-between samples, to generate the output sequence according to the relation

$$y[n] = x[nM]$$

(ii)

$$Y(z) = \sum_{n=-\infty}^{+\infty} x[nM] z^{-n}$$

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \dots, \\ 0, & \text{otherwise} \end{cases}$$

$$x_{\text{int}}[n] = x[n]$$

$$Y(z) = \sum_{n=-\infty}^{+\infty} x[nM] z^{-n} = \sum_{n=-\infty}^{+\infty} x_{\text{int}}[nM] z^{-n} = \sum_{k=-\infty}^{+\infty} x_{\text{int}}[k] z^{-k/M} = X_{\text{int}}(z^{1/M})$$

$$x_{\text{int}}[n] = c[n]x[n]$$

$$c[n]$$

$$c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \dots, \\ 0, & \text{otherwise} \end{cases}$$

$$W_M = e^{-\frac{j2\pi}{M}}$$

$$X_{\text{int}}(z) = \sum_{n=-\infty}^{+\infty} c[n]x[n]z^{-n} = \frac{1}{M} \sum_{n=-\infty}^{+\infty} \left( \sum_{k=0}^{M-1} W_M^{kn} \right) x[n]z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \left( \sum_{n=-\infty}^{+\infty} x[n]W_M^{kn} z^{-n} \right) = \frac{1}{M} \sum_{k=0}^{M-1} X(zW_M^{-k})$$

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k})$$

- (b) Determine the condition under which a cascade of a factor of  $M$  down sampler with a factor of  $L$  up sampler is interchangeable.

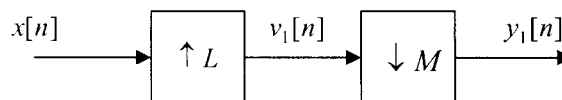


Figure 1

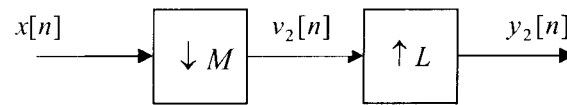


Figure 2

For Figure 1 we have  $V_1(z) = X(z^L)$  and  $Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-kL})$

For Figure 2 we have

$$V_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^{-k}) \text{ and } Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{-k})$$

The functions  $W_M^{-k}$  and  $W_M^{-kL}$  are the same only if the numbers  $L$  and  $M$  are relatively prime.