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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

PROBABILITY AND STOCHASTIC PROCESSES

Monday, 28 April 10:00 am

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : R.B. Vinter
Second Marker(s) : J.M.C. Clark

No special instructions for invigilators or instructions for candidates for this paper.

1. (a) Consider the network of Figure 1 involving 4 switches, labelled S_1, S_2, S_3 and S_4. Assume that events

$$A_i = \{\text{'}S_i \text{ is closed'}\}, \quad i = 1, \ldots, 4$$

are independent and, furthermore, that

$$P[A_i] = p,$$

for some constant p, $0 \leq p \leq 1$. Determine the probability that there is a closed path connecting the signal source at (a) and the signal destination at (b). [12]

(b) A signal comes from one and only one of 3 possible sources, labelled s_1, s_2 and s_3. The receiver indicates that either the signal source is s_1 or s_2 or s_3. For $i = 1, 2$ and 3 write

$$\begin{aligned} A_i &= \{\text{'the signal source is } s_i\text{'}\} \\ B_i &= \{\text{'the receiver indicates that the signal source is } s_i\text{'}\}. \end{aligned}$$

Assume that

$$P[A_1] = 0.9, \quad P[A_2] = 0.05 \text{ and } P[A_3] = 0.05.$$

Assume further that, for $i, j = 1, 2$ and 3, we have

$$P[B_i|A_j] = \begin{cases} 0.8 & \text{if } j = i \\ 0.1 & \text{if } j \neq i. \end{cases}$$

The receiver indicates that the signal source is s_1. Determine the probability that the signal source really is s_1. [8]

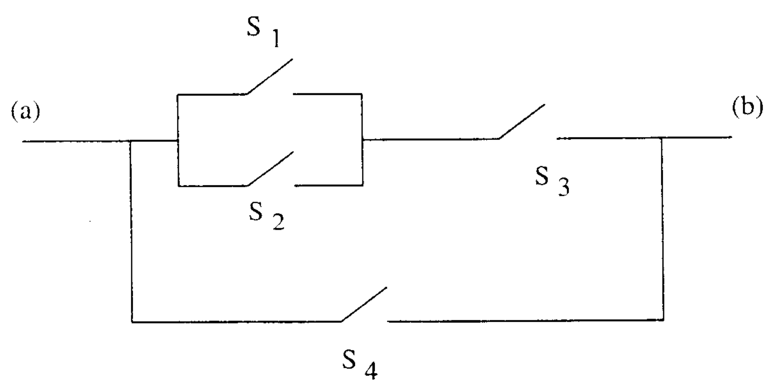

Figure 1

2 A signal $X(\omega)$ has uniform distribution

$$f_X(x) = \begin{cases} 0.5 & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise}. \end{cases}$$

A digitized, noise-free measurement $Y(\omega)$ is made of $X(\omega)$, namely:

$$Y(\omega) = \begin{cases} 0 & \text{if } 0 \le X(\omega) \le 1 \\ 1 & \text{if } 1 \le X(\omega) \le 2 \end{cases}$$

Determine the conditional probability distibution function $F_{X|Y}(x|y)$ of $X(\omega)$ given $Y(\omega) = y$, $y = 0, 1$. Show that it has a density $f_{X|Y}(x|y)$. [10]

Hence determine the least squares estimate $\hat{X}(y)$ of $X(\omega)$ given $Y(\omega) = y$, $y = 0, 1$. Evaluate the mean square error [4]

$$J_{\text{min}} = E[|X - \hat{X}(Y)|^2].$$

[4]

Is $\hat{X}(y)$ a *linear* estimator? [2]

Hint: When evaluating the mean square error, use the relationship

$$E[g(X,Y)] = \sum_{i=0}^{1} \int g(x,y) f_{X|Y}(x|y=i)dxP[Y(\omega)=i],$$

for any function $g(x, y)$.

3. (a) Let $T_1(\omega)$ and $T_2(\omega)$ be independent random variables that both have the probability density function

$$f(t) = \begin{cases} \frac{1}{T} & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise ,} \end{cases} \qquad (1)$$

for some constant $T > 0$. Derive the probability density function of $Y(\omega)$, where

$$Y(\omega) = T_1(\omega) + T_2(\omega) .$$

[10]

(b) A device is powered by a battery (labelled A) and a back-up battery (labelled B): when battery A is expended, battery B is switched in to replace it.

The use/charging history of the batteries are unknown, and so the periods of time $T_1(\omega)$ and $T_2(\omega)$ for which batteries A and B, respectively, can supply power are modelled as random variables. Furthermore the switch for the back-up battery may fail. Define the event

$$F = \{\text{'the switch for battery } B \text{ fails when battery } A \text{ is expended'}\}$$

and assume

(i): $T_1(\omega)$ and $T_2(\omega)$ both have the probability density function $f(t)$ defined in (1), for the same value of the design constant $T > 0$.

(ii): The random variables $T_1(\omega)$ and $T_2(\omega)$ and the event F are independent,

(iii): $P[F] = 0.1$.

Denote by $Z(\omega)$ the total operating time of the device. Derive the probability density function $f_Z(z)$ of $Z(\omega)$. It is required that [4]

$$P[\text{'the device is powered for at least 10 hours'}] \geq 0.95 .$$

What is the least value of the constant T to achieve this specification? [6]

4. (a) Take a random variable $D(\omega)$. Show that the function

$$V(d) = E[|D - d|^2]$$

is minimized at $d = E[D]$. [2]

(b) A surveillance system aims to estimate the midpoint of a randomly located vehicle from noisy measurements of the location of the front and back of the vehicle. Write

$$\begin{aligned} F(\omega) &= \text{location of front of vehicle,} \\ B(\omega) &= \text{location of back of vehicle,} \\ Y(\omega) &= \text{measurement of location of front of vehicle,} \\ Z(\omega) &= \text{measurement of location of back of vehicle.} \end{aligned}$$

Assume that, for some positive constants α^2 and σ^2 and $L > 0$ and some number r, $-1 < r < +1$,

$$\begin{aligned} &\text{cov}(F, Y) = \text{cov}(B, Z) = \alpha^2 \quad \text{and} \quad \text{cov}(F, Z) = \text{cov}(B, Y) = 0\,, \\ &\text{var}(Y) = \text{var}(Z) = \sigma^2 \quad \text{and} \quad \text{cov}(Y, Z) = r\sigma^2\,, \\ &E[B] = E[Z] = 0 \quad \text{and} \quad E[F] = E[Y] = L\,. \end{aligned}$$

Using part (a), or otherwise, show that the linear least squares estimate $\hat{X}(\omega)$ of the midpoint,

$$X(\omega) = \frac{1}{2}[F(\omega) + B(\omega)]\,,$$

given $Y(\omega)$ and $Z(\omega)$, is

$$\hat{X}(\omega) = a^* Y(\omega) + b^* Z(\omega) + c^*\,,$$

where a^*, b^* and c^* are chosen to be

$$a^* = b^* = \frac{\alpha^2}{2\sigma^2(1+r)} \quad \text{and} \quad c^* = (\frac{1}{2} - a^*) \times L.$$

[16]

Now suppose that α^2 and σ^2 are design constants, but r is a sensor design parameter which can be varied in the range $-1 < r < 1$. Without doing any calculations, suggest the value of design parameter r for which the mean square estimation error of the above linear estimator is minimized. [2]

5. (a) Consider the stationary n-vector process $\{x_k\}$ generated by the equations

$$x_{k+1} = Ax_k + be_k .$$

Here, A is a given $n \times n$ matrix and b is a given n-vector. $\{e_k\}$ is a sequence of zero mean, uncorrelated random variables, each with variance σ^2. Write

$$R_x(0) = E[x_k x_k^T] ,$$

Show that $R_x(0)$ satisfies the matrix equation

$$R_x(0) = AR_x(0)A^T + \sigma^2 bb^T .$$

[8]

(b) Two coupled stationary scalar processes $\{y_k\}$ and $\{v_k\}$ are generated by the equations

$$\begin{aligned} y_{k+1} &= 0.5y_k + \alpha v_k \\ v_{k+1} &= 0.2v_k + e_k , \end{aligned}$$

in which $\{e_k\}$ is a sequence of uncorrelated, zero mean random variables, each with variance σ^2. The coupling coefficient $\alpha > 0$ is an unknown positive constant.

It is known that

$$E[y_k^2] = 2\,E[v_k^2] .$$

Using the results of part (a), or otherwise, determine α. [12]

6. (a) Define the spectral density $\Phi_x(\omega)$ of a stationary, second order, zero mean, scalar stochastic process $\{x_k\}$. [2]

Now suppose that the stationary, second order, zero mean, scalar stochastic process $\{y_k\}$ is related to $\{x_k\}$ according to the difference equation

$$y_k = a_0 x_k + a_1 x_{k-1} \,.$$

Here, a_0 and a_1 are constants. Show that the spectral density $\Phi_y(\omega)$ of $\{y_k\}$ is given by

$$\Phi_y(\omega) = F(e^{j\omega})F(e^{-j\omega})\Phi_x(\omega) \,.$$

Here, $F(z) = a_0 + a_1 z^{-1}$. [6]

(b) A signal $\{v_k\}$ is modelled as the output of a linear system with unknown rational transfer function $D(z)$ driven by a sequence of zero mean, uncorrelated, unit variance random variables $\{e_k\}$. See Figure 6. It is not possible to measure $\{v_k\}$ directly. Measurements are available however of a process $\{y_k\}$ that is v_k contaminated by an 'echo':

$$y_k = v_k + a v_{k-d} \,,$$

where a is a given positive number (the relative magnitude of the echo) and d is a positive integer (the echo delay). The spectral density of $\{y_k\}$ is known to be

$$\Phi_y(\omega) = \frac{\frac{17}{16} + \frac{1}{2}\cos(2\omega)}{(\frac{5}{4} + \cos(\omega))(\frac{9}{10} + \frac{2}{3}\cos(\omega))} \,.$$

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Determine the values of the constants a and d. ~~Develop an ARMA~~ model for the process $\{v_k\}$ consistent with this spectral density. [12]

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G.C.

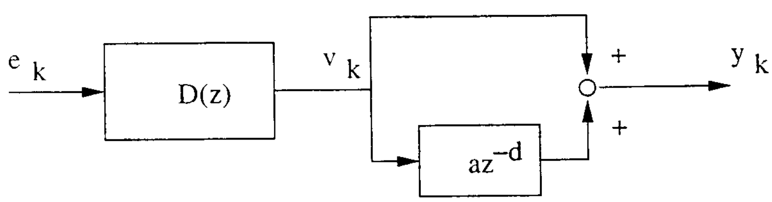

Figure 6