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C2.1
SC4

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE PART IV: M.Eng. and ACGI

PROBABILITY AND STOCHASTIC PROCESSES

Wednesday, 9 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

4, 5, 6

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Special instructions for invigilators: None

Information for candidates: None

- 1(a) A source signal X and a transmitted signal Z are modelled as binary random variables taking values 0 and 1.

A received signal Y is modelled as

$$Y = NZ$$

in which N is a binary random variable which takes values 0 or 1 and is independent of X and Z . ($N = 0$ corresponds to receiver failure.)

Assume that

$$P[X = 0] = P[X = 1] = 1/2$$

$$P[Z = 0|X = 0] = P[Z = 1|X = 1] = \alpha$$

$$P[N = 1] = p.$$

A signal $Y = 0$ is received. What is the probability that the source signal was $X = 0$?

- 1(b) A section of a communication link is modelled as indicated in *Figure 1*. The possible states of each switch S_1, \dots, S_5 are *open* and *closed*. The events

$$S_i \text{ is closed} \quad i = 1, \dots, 5$$

are independent and

$$P[S_i \text{ is closed}] = q \quad \text{for each } i.$$

Calculate the probability that there is a closed path between points A and B .

Now suppose that a communication link is composed of identical sections, connected in series. Suppose that the states of all switches are independent,

$$q = (1 - 10^{-3})$$

and that the length of each section is 1 metre. What is the maximum length of the communication link such that

$$P[\text{there is a closed path through the entire link}] \geq 0.99?$$

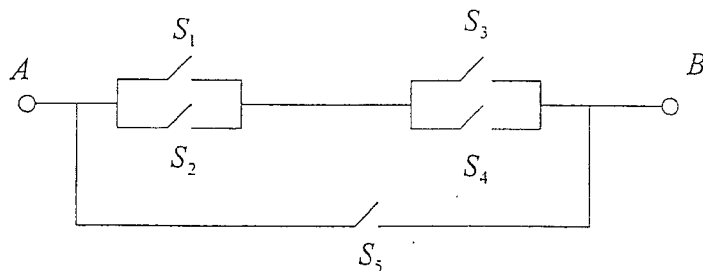


Figure 1

2 A signal $X(\omega)$, corrupted by a bias n , is rectified by a device with characteristic

$$R(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

$X(\omega)$ is modelled as a random variable with uniform probability density function

$$f_X(x) = \begin{cases} 1/2 & \text{for } -1 \leq x \leq +1 \\ 0 & \text{otherwise} \end{cases}.$$

The bias n is taken to be a constant ($0 \leq n \leq 1$). See *Figure 2*.

Write $Z(\omega)$ and $Z^*(\omega)$ for the rectified noisy signal and the rectified noise-free signal respectively:

$$\begin{aligned} Z(\omega) &= R(X(\omega) + n) \\ Z^*(\omega) &= R(X(\omega)). \end{aligned}$$

Calculate the average power of the rectifier output error, namely

$$E[(Z(\omega) - Z^*(\omega))^2].$$

(Assume n is small, so that terms involving n^3 , n^4 , etc., can be ignored).

Hence determine the signal to noise ratios S_{in} and S_{out} at the input and output of the rectifier respectively

$$S_{in} = \frac{n^2}{E[X^2(\omega)]} \quad \text{and} \quad S_{out} = \frac{E[(Z(\omega) - Z^*(\omega))^2]}{E[Z^{*2}(\omega)]}.$$

Notice that $S_{in} \neq S_{out}$. Is this a nonlinear phenomenon, i.e., is it possible that $S_{in} \neq S_{out}$ if the nonlinear characteristic $R(x)$ were replaced by a pure gain Kx ($K \neq 0$)?

Hint: obtain formulae for $R(x+n) - R(x)$ in the regions

$$(i) \ x < -n \quad (ii) \ -n \leq x \leq 0 \quad (iii) \ x \geq 0.$$

Do not attempt to calculate the distribution function of $Z(\omega)$.

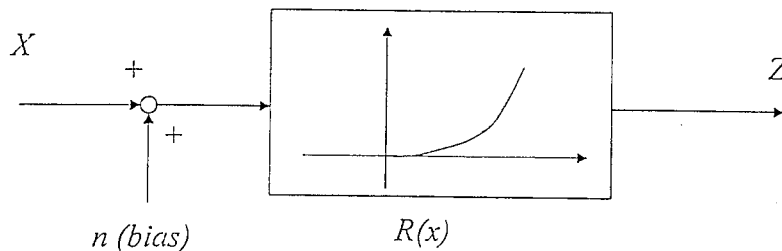


Figure 2

- 3(a) Let $X(\omega)$ and $Y(\omega)$ be jointly distributed, zero mean, normal random variables, with variance σ^2 and correlation coefficient r , $-1 < r < +1$:

$$f_{XY}(x, y) = \frac{1}{\sqrt{\pi(1-r^2)}\frac{1}{2}\sigma^2} \exp \left\{ -\frac{1}{2\sigma^2(1-r^2)} (|x - ry|^2 + (1-r^2)y^2) \right\}.$$

Derive the conditional probability density $f_{X|Y}(x; y)$ of $X(\omega)$ given $Y(\omega) = y$. Show that the mean and variance of the conditional probability density are

$$\begin{aligned} m_{X|Y}(y) &= ry, \\ \text{var}_{X|Y}(y) &= \sigma^2(1-r^2). \end{aligned}$$

- 3(b) Now interpret $X(\omega)$ as a signal and $Y(\omega)$ as a measurement of the signal and

$$\hat{X}(y) = m_{X|Y}(y)$$

as an estimator of $X(\omega)$ given $Y(\omega) = y$.

The higher the quality of the sensor, the larger is the absolute value of the correlation coefficient. Assume

$$\text{sensor cost} = \frac{10}{1-|r|^2} \text{ (\$'s)}$$

What is the minimum cost of the sensor, if we require

$$\text{var}_{X|Y}(y) \leq 0.01E[|X(\omega)|^2] \quad ?$$

(In part (a), you can use the fact that

$$f(x) = \frac{1}{(\sqrt{2\pi}\sigma)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2\sigma^2}(x-m)^2 \right\}$$

is the probability density function of a random variable with mean m and variance σ^2 .)

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4 A transmitted signal $X(\omega)$ is modelled as a continuous random variable with probability density function $f_X(x)$. The received signal $Y(\omega)$ has a random offset $N(\omega)$:

$$Y(\omega) = X(\omega) + N(\omega).$$

The offset is modelled as a discrete random variable, independent of $X(\omega)$, with

$$P[N(\omega) = 0] = p \quad \text{and} \quad P[N(\omega) = \frac{1}{2}] = (1 - p).$$

Show that the conditional probability mass function of $X(\omega)$ given $Y(\omega) = y$ is

$$P[X(\omega) = y | Y(\omega) = y] = \frac{pf_X(y)}{pf_X(y) + (1-p)f_X(y - \frac{1}{2})},$$

$$P[X(\omega) = y - \frac{1}{2} | Y(\omega) = y] = \frac{(1-p)f_X(y - \frac{1}{2})}{pf_X(y) + (1-p)f_X(y - \frac{1}{2})}.$$

Hence derive a formula for the least squares estimate

$$\hat{X} = g(y)$$

of $X(\omega)$ given $Y(\omega) = y$.

Evaluate $g(y)$ in the case when

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Comment on the results.

Hint: In the last part of the question, consider separately the cases

$$(i) 0 \leq y < \frac{1}{2}, \quad (ii) \frac{1}{2} \leq y \leq 1, \quad (iii) 1 < y \leq 1\frac{1}{2}$$

- 5(a) An r -vector stationary stochastic process $\{y_k\}$ is modelled by the state space equations

$$x_{k+1} = Ax_k + be_k \quad y_k = Cx_k$$

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in which A is a 'stable' $n \times n$ matrix, b is an n -vector, C is an $r \times n$ matrix and $\{e_k\}$ is a sequence of zero mean, independent scalar random variables with common variance

$$E[e_k^2] = \sigma^2.$$

Derive a set of equations, including the matrix Lyapunov equation, for the covariance function $R_y(l)$, $l = \dots, -1, 0, +1, \dots$

- 5(b) Consider the two stationary scalar processes $\{u_k\}$ and $\{v_k\}$ modelled by the difference equations:

$$\begin{aligned} u_{k+1} &= v_k + e_k \\ v_{k+1} &= -au_k + e_k. \end{aligned}$$

Here, a is a modelling parameter and $\{e_k\}$ is a sequence of zero mean, independent scalar random variables with common variance

$$E[e_k^2] = \sigma^2.$$

It is known that the covariance functions $R_u(l)$ and $R_v(l)$ of $\{u_k\}$ and $\{v_k\}$ satisfy

$$R_v(0) = \frac{5}{8}R_u(0).$$

By using the results of Part (a), or otherwise, determine the value of a .

6(a) Define the spectral density $\Phi_r(\omega)$ of a stationary, second order, zero mean, scalar stochastic process $\{r_k\}$.

Now suppose that

$$r_k = u_k + v_k, \quad k = \dots, -1, 0, +1, \dots$$

for stationary, second order, zero mean, scalar stochastic processes $\{u_k\}$ and $\{v_k\}$ such that

$$u_k \text{ and } v_j \text{ are independent for all } k, j.$$

Show that

$$\Phi_r(\omega) = \Phi_u(\omega) + \Phi_v(\omega).$$

6(b) A transmitted signal $\{s_k\}$ is modelled as a stationary, zero mean, scalar stochastic process satisfying

$$(2 - z^{-1})s_k = e_k,$$

in which $\{e_k\}$ is a scalar, unit variance, white noise process.

Due to the presence of noise and channel distortion, the received signal $\{r_k\}$ is the solution to the difference equation

$$(3 - z^{-1})r_k = s_k + d_k.$$

Here, the disturbance process $\{d_k\}$ is generated by the equation

$$d_k = D(z)\tilde{e}_k,$$

in which $D(z)$ is an unknown transfer function and $\{\tilde{e}_k\}$ is a unit variance, white noise process, independent of $\{e_k\}$. See *Figure 3*.

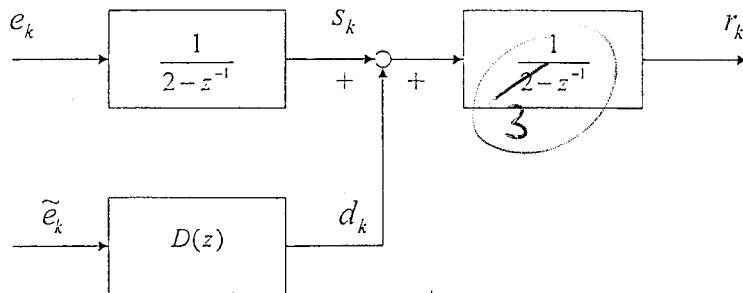
The spectral density of the received signal $\{r_k\}$ is known to be

$$\Phi_r(\omega) = \frac{1 + (17 - 4e^{j\omega} - 4e^{-j\omega})(5 - 2e^{j\omega} - 2e^{-j\omega})}{(5 - 2e^{j\omega} - 2e^{-j\omega})(10 - 3e^{j\omega} - 3e^{-j\omega})}.$$

Find the transfer functions $G_1(z)$ and $G_2(z)$ such that

$$r_k = u_k + v_k, \text{ where } u_k = G_1 e_k, v_k = G_2 \tilde{e}_k.$$

By using the results of Part (a), or otherwise, determine the transfer function $D(z)$ which generates the disturbance process $\{d_k\}$.



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Figure 3

Prob. & Stoch. Processes Exam, 2001. Model Answers

$$\begin{aligned}
 1 \text{ (i)} \quad P[X=0|Y=0] &= \frac{P[X=0 \& Y=0]}{P[Y=0]} = \frac{P[X=0 \& Y=0|N=0] \cdot P[N=0]}{P[Y=0|N=0]P[N=0] + P[Y=0|N=1]P[N=1]} \\
 &= \frac{P[X=0]P[N=0] + P[X=0 \& Z=0] \cdot P[N=1]}{P[Y=0|N=0]P[N=0] + P[Y=0|N=1]P[N=1]} \\
 &= \frac{P[X=0]P[N=0] + P[Z=0|X=0]P[X=0] \cdot P[N=1]}{P[N=0] + P[Z=0]P[N=1]} = \frac{\frac{1}{2}(1-p) + \frac{1}{2} \times p}{(1-p) + P[Z=0]p}
 \end{aligned}$$

$$\begin{aligned}
 \text{But } P[Z=0] &= P[Z=0|X=0]P[X=0] + P[Z=0|X=1]P[X=1] \\
 &= \alpha \frac{1}{2} + (1-\alpha) \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

$$\underline{8} \quad \text{Hence } P[X=0|Y=0] = \frac{\frac{1}{2}(1-p) + \frac{1}{2} \times p}{1-p + \frac{1}{2}p} = \frac{1 - (1-\alpha)p}{2-p}$$

(ii) The event "there is a closed path between A and B" is

$$Q = [(S_1 \cup S_2) \cap (S_3 \cup S_4)] \cup S_5. \quad \text{Write } \bar{Q} = \text{"complement of } Q \text{" etc.}$$

$$\text{Then } \bar{Q} = \bar{S}_5 \cap (\bar{S}_1 \cup \bar{S}_2) \cap (\bar{S}_3 \cap \bar{S}_4) \quad (\text{Here } S_i = \{S_i \text{ is closed}\}, \text{ etc.})$$

But

$$\begin{aligned}
 P[(S_1 \cup S_2) \cap (S_3 \cap S_4)] &= P[S_1 \cup S_2] \cdot P[S_3 \cap S_4] \quad (\text{by independence}) \\
 &= (1 - P[\bar{S}_1 \cap \bar{S}_2]) (1 - P[\bar{S}_3 \cap \bar{S}_4]) \\
 &= (1 - P[\bar{S}_1] \cdot P[\bar{S}_2]) (1 - P[\bar{S}_3] \cdot P[\bar{S}_4]) \quad (\text{by independence}) \\
 &= (1 - (1-q)^2)^2
 \end{aligned}$$

$$\text{Hence } P[(S_1 \cup S_2) \cap (S_3 \cap S_4)] = 1 - (1 - (1-q)^2)^2. \quad \text{It follows that}$$

$$P[Q] = 1 - P[\bar{Q}] = 1 - P[\bar{S}_5] (1 - (1 - (1-q)^2)^2) \quad (\text{by independence})$$

Then

$$\begin{aligned}
 P(Q) &= 1 - (1-q) (1 - (1-q)^2 + 2(1-q)^2 - (1-q)^4) \\
 &= 1 - (1-q)^3 (2 - (1-q)^2)
 \end{aligned}$$

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$$\text{If } 1-q = 10^{-6}, \quad P(Q) = 1 - 2 \times 10^{-9}$$

If N is length of link, we require

$$(1 - 2 \times 10^{-9})^N \geq 0.99$$

$$\text{or } 1 - 2 \times N \times 10^{-9} \geq 0.99$$

(by independence)
(approximately)

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$$\text{or } N \geq \frac{1}{2} \times 10^9 \times 10^{-2} = \frac{1}{2} \times 10^7 \text{ meters}$$

$$2. \quad R(x+n) - R(x) = \begin{cases} (x+n)^2 - x^2 & x \geq 0 \\ (x+n)^2 & -n \leq x < 0 \\ 0 & x < -n \end{cases}$$

The power of the rectified output is therefore

$$\begin{aligned} E[|R(x(n)) - R(x)|^2] f_X(x) dx &= \frac{1}{2} \int_{-1}^1 (R(x+n) - R(x))^2 f_X(x) dx \\ &= \frac{1}{2} \int_0^1 [(x+n)^4 - 2(x+n)^2 x^2 + x^4] dx + \frac{1}{2} \int_{-n}^0 (x+n)^4 dx \\ &= \frac{1}{2} \int_0^1 [(x+n)^4 - 2x^4 - 4x^3 n - 2x^2 n^2 + x^4] dx + \frac{1}{2} \int_{-n}^0 (x+n)^4 dx \\ &= \frac{1}{2} \left\{ \frac{1}{5} (x+n)^5 - \frac{2}{5} x^5 - x^4 n - \frac{2}{3} x^3 n^2 + \frac{1}{5} x^5 \Big|_0^1 + \frac{1}{5} n^5 \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{5} (1+n)^5 - \frac{1}{5} n^5 - \frac{2}{5} - n - \frac{2}{3} n^2 + \frac{1}{5} + \frac{1}{5} n^5 \right\} \quad \leftarrow \begin{array}{l} \text{o's for terms} \\ \text{w/ } n^3, n^4, \dots \end{array} \\ &\approx \frac{1}{2} \left\{ \frac{1}{5} [1+5n+10n^2] - \frac{2}{5} - n - \frac{2}{3} n^2 + \frac{1}{5} + 0 + 0 \right\} \\ 14 \quad &= \frac{1}{2} \times \left(2 - \frac{2}{3} \right) n^2 = \underline{\underline{\frac{2}{3} n^2}} \end{aligned}$$

Also

$$E[X^2] = \frac{1}{2} \int_{-1}^1 x^2 dx = \frac{1}{2} \cdot \frac{x^3}{3} \Big|_{-1}^1 = \frac{1}{3}$$

$$E[Z^2] = \frac{1}{2} \int_0^1 x^4 dx = \frac{1}{10}$$

It follows that (for n small)

$$S_{in} = \frac{n^2}{\frac{1}{3}} = \frac{3n^2}{1} \quad \text{and} \quad S_{out} \approx \underline{\underline{\frac{20}{3} n^2}}$$

2 We see that $S_{in} \neq S_{out}$.

If $R(x)$ is replaced by a linear gain, $z - z^* = K(x+n) - Kx = Kn$
and $z^* = Kx$. So

$$S_{out} = \frac{K^2 n^2}{K^2 E[|X|^2]} = \frac{n^2}{E[|X|^2]} = S_{in}$$

4 We have confirmed that $S_{in} \neq S_{out}$ is a nonlinear phenomenon.

$$3(i) f_Y(y) = (\pi[1-r^2] \frac{1}{2} \sigma^2)^{-1/2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{|x-ry|^2}{2\sigma^2(1-r^2)}\right\} dx \cdot \exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$

change variables $x' = x - ry$ (for fixed y), giving

$$\int_{-\infty}^{+\infty} \exp\{\dots\} = \int_{-\infty}^{+\infty} \exp\left\{-\frac{x'^2}{2\sigma^2(1-r^2)}\right\} dx'$$

= $\pi^{1/2} \sqrt{\sigma^2(1-r^2)}$ (by properties of normal density)

Hence $f_Y(y) = (\pi\sigma^2)^{-1/2} \exp\left\{-\frac{y^2}{2\sigma^2}\right\}$. So

$$f_{X|Y}(x;y) = \frac{(\pi\sigma^2)^{1/2}}{\pi(1-r^2)^{1/2}\sigma^2} \cdot \exp\left\{-\frac{1}{2\sigma^2} a\right\} \quad \text{where}$$

$$a = \frac{1}{1-r^2} \left(|x-ry| + (1-r^2)y^2 \right) - y^2$$

$$= \frac{1}{(1-r^2)} \left(x^2 - 2ryx + r^2y^2 + y^2 - r^2y^2 - y^2 + r^2y^2 \right) = \frac{|x-ry|^2}{1-r^2}$$

We have shown

$$f_{X|Y}(x;y) = \frac{1}{(\pi\bar{\sigma}^2)^{1/2}} \exp\left\{-\frac{1}{2\bar{\sigma}^2} (x-\bar{m})^2\right\}$$

where $\bar{m} (= m_{X|Y}(y)) = ry$ and $\bar{\sigma}^2 (= \text{var}_{X|Y}(y)) = \sigma^2(1-r^2)$

(ii) We require that

$$\text{var}_{X|Y}(y) \leq 0.01 E|X|^2$$

But $E|X|^2 = \sigma^2$ and $\text{var}_{X|Y}(y) = \sigma^2(1-r^2)$

Hence

$$\sigma^2(1-r^2) \leq 0.01 \sigma^2$$

It follows $\text{cost} = \frac{10}{1-r^2} \geq 1000$ (£'s)

6 So minimum cost (to achieve specifications) is £1,000

4 If $Y(\omega) = y$ then $X(\omega)$ can only take values y and $y - \frac{1}{2}$.

For $\beta = 0$ or $\frac{1}{2}$

$$m_{X|Y}(X = y - \beta | Y = y) \approx \frac{P[y \leq X + \beta \leq y + \delta_y | y \leq Y \leq y + \delta_y]}{P[y \leq X + N \leq y + \delta_y | N=0]P[N=0] + P[y \leq X + N \leq y + \delta_y | N=\frac{1}{2}]P[N=\frac{1}{2}]}$$

(by Bayes' Rule)

$$= \frac{P[N = \beta] \cdot f_X(y - \beta)}{[P_X(y)P + f_X(y - \frac{1}{2})(1-P)]}$$

It follows

$$m_{X|Y}(X = x | Y = y) = \begin{cases} \frac{\alpha_1}{\alpha_1 + \alpha_2} & \text{if } x = y \\ \frac{\alpha_2}{\alpha_1 + \alpha_2} & \text{if } x = y - \frac{1}{2} \end{cases}$$

where $\alpha_1 = p f_X(y)$ and $\alpha_2 = (1-p) f_X(y - \frac{1}{2})$

Hence, the least squares estimate, which coincides with the conditional mean, is

$$g(y) = \frac{\alpha_1 y + \alpha_2 (y - \frac{1}{2})}{\alpha_1 + \alpha_2}$$

If $f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$, then

for $0 \leq y < \frac{1}{2}$, $\alpha_1 = p$, $\alpha_2 = 0$, so $g(y) = y$

for $\frac{1}{2} \leq y \leq 1$, $\alpha_1 = p$, $\alpha_2 = (1-p)$, so $g(y) = py + (1-p)(y - \frac{1}{2})$

for $1 < y \leq 1\frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = (1-p)$ so $g(y) = y - \frac{1}{2}$

The point here is that, if $y < \frac{1}{2}$, then we must have $N=0$, and $X(\omega) = Y(\omega)$ so $g(y) = y$ is the best estimate of $X(\omega)$ given $Y(\omega) = y$.

If $y > 1$, we must have $N = \frac{1}{2}$, so $X(\omega) = Y(\omega) - \frac{1}{2}$.

Here $g(y) = y - \frac{1}{2}$ is the best estimate of $X(\omega)$

For $\frac{1}{2} \leq y \leq 1$, $g(y)$ is close to y if $N(\omega) = 0$ with high probability
 $g(y)$ is close to $y - \frac{1}{2}$ if $N(\omega) = \frac{1}{2}$ with high probability

5. (a) We have $x_{k+1} = Ax_k + be_k$. It follows that

$$x_{k+1}x_{k+1}^T = (Ax_k + be_k)(Ax_k + be_k)^T$$

But, from the state equations, x_k is independent of e_k . Hence

$$E\{be_kx_k^Tb^T\} = 0$$

$$\text{Hence } E\{x_{k+1}x_{k+1}^T\} = E\{Ax_kx_k^TA^T\} + 0 + 0 = E\{be_kx_kx_k^Tb^T\}$$

It follows that

$$R_x(0) = AR_x(0)A^T + \sigma^2bb^T \quad \text{--- (1)}$$

Also $x_{k+1}x_k^T = Ax_kx_k^T + be_kx_k^T$. Taking expectations gives

$$R_x(1) = AR_x(0)$$

Similarly $R_x(l) = A^l R_x(0)$ for $l=1, 2, \dots$

and $R_x(l-1) = R_x(0), A^{l-1}$ for $l=1, 2, \dots$

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Then $R_y(l) = C E\{x_kx_{k-l}^T\} C^T = CR_x(l)C^T$, for $l=1, 2, \dots$

(b) We can regard u_k and v_k as components of the 2-vector process

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e_k$$

Write $R_x(0) = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$. Then, by (1),

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -a \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \sigma^2$$

$$\Rightarrow \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} r_{22} & -ar_{21} \\ -ar_{12} & r_{11} + a^2r_{22} \end{bmatrix} + \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Equating matrix entries, we obtain

$$r_{11} = r_{22} + \sigma^2, \quad r_{12} = -ar_{21} + \sigma^2, \quad r_{21} = -ar_{12} + \sigma^2, \quad r_{22} = a^2r_{11} + \sigma^2$$

Hence

$$r_{11} = a^2r_{11} + 2\sigma^2 \Rightarrow r_{11} = \frac{2\sigma^2}{1-a^2}$$

$$r_{22} = \sigma^2 \frac{1+a^2}{1-a^2}$$

Since $r_{11} = R_u(0)$ and $r_{22} = R_v(0)$, we have

$$\frac{5}{8} = R_v(0)/R_u(0) = r_{22}/r_{11} = \frac{1}{2}(1+a^2)$$

12 Hence $a^2 = \left(\frac{5}{4} - 1\right) \Rightarrow a = \pm \frac{1}{2}$

6(i) The spectral density $\Phi_r(\omega)$ of $\{r_k\}$ is

$$\Phi_r(\omega) = \sum_{l=-\infty}^{\infty} R(l) e^{j\omega l}, \text{ where } R(l) = E\{r_k r_{k-l}\}.$$

If $r_k = u_k + v_k$,

$$\begin{aligned} R_r(l) &= E\{(u_k + v_k)(u_{k-l} + v_{k-l})\} \\ &= E\{u_k u_{k-l}\} + 0 + 0 + E\{v_k v_{k-l}\} \\ &= R_u(l) + R_v(l). \end{aligned}$$

6 It follows that $\Phi_r(\omega) = \sum_{l=-\infty}^{\infty} [R_u(l) + R_v(l)] e^{j\omega l} = \Phi_u(\omega) + \Phi_v(\omega)$.

(ii) $(2-z^{-1})s_k = e_k$, $(3-z^{-1})r_k = s_k + d_k$, $d_k = N(z)\tilde{e}_k$ imply
 $r_k = \frac{1}{3-z^{-1}} \left(\frac{e_k}{(2-z^{-1})} + D(z)\tilde{e}_k \right) = G_1(z)e_k + G_2(z)\tilde{e}_k$

in which $G_1(z) = \frac{1}{(2-z^{-1})(3-z^{-1})}$ and $G_2(z) = \frac{D(z)}{(3-z^{-1})}$

We know:

$$\Phi_r(\omega) = \frac{1 + (17 - 4e^{j\omega} - 4e^{-j\omega})(5 - 2e^{j\omega} - 2e^{-j\omega})}{(5 - 2e^{j\omega} - 2e^{-j\omega})(10 - 3e^{j\omega} - 3e^{-j\omega})}$$

But, writing $u_k = G_1(z)e_k$, we have

$$\Phi_u(\omega) = \frac{1}{(2-z^{-1})(2-z)(3-z^{-1})(3-z)} \Big|_{z=e^{j\omega}} = \frac{1}{(5-2e^{j\omega}-2e^{-j\omega})(10-3e^{j\omega}-3e^{-j\omega})}$$

So, writing $v_k = G_2(z)e_k$, we have

$$\begin{aligned} \Phi_v(\omega) &= \Phi_r(\omega) - \Phi_u(\omega) = \frac{(17-4e^{j\omega}-4e^{-j\omega})(5-2e^{j\omega}-2e^{-j\omega})}{(5-2e^{j\omega}-2e^{-j\omega})(10-3e^{j\omega}-3e^{-j\omega})} \\ &= \frac{(4-z^{-1})}{(3-z^{-1})} \cdot \frac{(4-z)}{(3-z)} \Big|_{z=e^{j\omega}} = G_2(z)G_2(z^{-1}) \Big|_{z=e^{j\omega}} \end{aligned}$$

It follows we can choose $G_2(z) = \frac{4-z^{-1}}{3-z^{-1}}$

Then $D(z) = (3-z^{-1})G_2(z) = \frac{4-z^{-1}}{3-z^{-1}}$

14 (Note: we cannot choose $G_2(z) = \frac{1-4z^{-1}}{1-3z^{-1}}$, because then $D(z) = \frac{(3-z^{-1})(1-4z^{-1})}{(1-3z^{-1})}$ is an unstable transfer function.