Paper Number(s): **E4.05**
SO7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

MSc and EEE PART IV: M.Eng. and ACGI

TRAFFIC THEORY & QUEUEING SYSTEMS

Tuesday, 22 May 10:00 am

There are FIVE questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Corrected Copy

Examiners: Barria,J.A. and De Wilde,P.

Special Information for Invigilators: NIL

Information for Candidates: NIL

1. (a) Derive the equilibrium equations for the Erlang model when the number of channels N is infinite ($N = \infty$); and when $N < \infty$. In the case of finite buffer ($N < \infty$) derive the probability of link saturation. State and explain any assumption made.

[10]

(b) Describe the interrupted Poisson process (IPP). Explain how this model can be used for the analysis of traffic offered to an overflow link. Use Markov chains if necessary.

[10]

2. (a) Explain what is meant by the balance equations of a stationary, continuous-time Markov chain and discuss the relationship between global balance and local balance. How is the property of reversibility related to the validity of the balance equations ?

[10]

(b) A circuit-switched telephone network is to be operated with automatic alternative routing subject to a trunk reservation constraint. Suppose that one of the links in the network consists of N channels operating with a trunk reservation parameter m: this means that first-choice traffic is accepted by the link whenever there is at least one free channel, but second-choice (i.e. re-routed) traffic is accepted only when the number of free channels is greater than m (where $m < N$).

Assuming that call holding times are exponential with a mean holding time of h, and that the first-choice and second-choice traffic streams can be regarded as independent Poisson streams with mean rates λ_1 *and* λ_2 respectively, set up a birth-death model for the total traffic carried on the link. Draw the state transition diagram for your model and write down the equilibrium equations for the system. Indicate briefly (full detail not required) how you would compute the blocking probabilities for the first-choice and second-choice in terms of the link idle probability π_0.

[10]

3. (a) Show that the mean waiting time for an *M/G/1* system is

$$E(w) = \left[\frac{\lambda E(S^2)}{2(1-\rho)}\right]$$

(i) State the meaning of λ, S, ρ.

[3]

(ii) State clearly all intermediate steps and any assumption made.

[7]

(b) A single-channel communication link is used for transmitting data files from one computer to another in a low-rate data network. The file length can be assumed to be exponentially-distributed with a mean file length of 700 kbytes and files arrive for transmission in a Poisson stream with a mean rate of 1 file/100 secs. The link is buffered by a FIFO buffer of sufficient capacity to hold all files awaiting transmission. If the channel transmission rate is 64 kbits/second:

(i) Determine the probability that a file will not have to wait for transmission

[4]

(ii) Determine the probability that the file will have to wait for more than 10 minutes before being transmitted

[4]

(iii) Would the overall mean waiting time be improved by giving "short" files (e.g. files less than 700 kbytes in length) non-pre-emptive priority over "long" files?. A brief discussion will be sufficient.

[2]

4. (a) For an N-voice source multiplexer, and using a Markov modulated Poisson process (MMPP) as your aggregated traffic model, obtain:

(i) The conditions on the maximum capacity of the multiplexer (assume service time distribution to be exponential with mean = *1/v*)

[5]

(ii) The state space representation (or Markov chain).

[5]

(b) Derive an approximated model for an ATM leaky bucket policy scheme. Discuss any underlying assumptions made.

[10]

5. (a) A system with $C = 8$ resource units is offered a mixture of Poisson traffic λ_1 requiring $b_1 = 1$ resource units and Poisson traffic λ_2 requiring $b_2 = 2$ resource units. The resources b_k ($k = 1,2$) holding time is exponentially distributed ($1/\mu_k$).

(i) State the expression and represent the state space, S, of the system on a two dimensional graph

[2]

(ii) State the expressions and identify the admission set, S_k, for traffic class k, $k = 1,2$

[2]

(iii) State the expressions for the blocking probability, B_k, for traffic class k, $k = 1,2$

[2]

(iv) State the expression of the steady state distribution of the system.

[2]

(b) A well known measure of Equivalence Capacity is given by the following expression:

$$C_l = \min[C_{ls}, C_{lf}]$$

where C_{lf} is a fluid-flow approximation of Equivalent Capacity and C_{ls} is the stationary approximation of the equivalent Capacity.

(i) State the underlying assumptions and models used to derive C_{lf} and C_{ls}

[4]

(ii) Explain the relevance of the Normal distribution approximation and main points in the derivation of C_{ls}

[4]

(iii) Explain the relevance of the survivor function and the main points in the derivation of C_{lf}.

[4]

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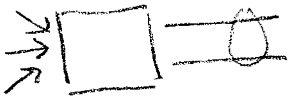
Q1 (a) Erlang Model

Traffic Sources



N-channels links

Access switch

Assumptions
- Total arrival stream: Poisson process (λ)
- channel holding time: Exponential r.v. ($1/\mu$)
- Full availability

N_t = nr. busy channels on link at time t

$\{N_t\}$ is a B/D process with:

(i) Birth coeff (for Poisson process)

$P[N_{t+\Delta t} = i+1 \mid N_t = i] = \lambda \Delta t \quad , i < N$

$\lambda_i = \lambda \quad \forall i < N$

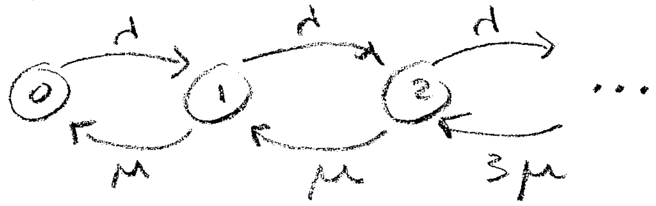
(ii) Death coeff
- if i channels are busy at time t for each of these

$P[\text{busy} \to \text{free}] = \mu \Delta t$
- channels act independently, the probability that exactly k channels will become idle in $(t, t+\Delta t)$ is binomial:

$P[k \text{ ch} \to \text{Idle}] = \binom{i}{k} (\mu \Delta t)^k (1-\mu\Delta t)^{i-k}$

$P[1 \text{ ch} \to \text{Idle}] = i\mu\Delta t + o(\Delta t)$

$\mu_i = i\mu \quad \forall i > 0$



Q1 (a)

(i) $N = \infty$

$$S = 1 + \frac{\lambda}{\mu} + \frac{\lambda}{\mu}\frac{\lambda}{2\mu} + \cdots = e^{\rho} < \infty$$

$$\rho = \frac{\lambda}{\mu}$$

Equilibrium equations

$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\pi_{i-1} = \left(\frac{\lambda}{i\mu}\right)\pi_{i-1} = \left(\frac{\rho}{i}\right)\pi_{i-1}$$

Recursive solution:

$$\pi_i = \left(\frac{\rho^i}{i!}\right)\pi_0 \qquad i = 1, 2, \ldots$$

$$\pi_0 = \frac{1}{S} = e^{-\rho}$$

$$\pi_i = \frac{\rho^i}{i!} e^{-\rho} \qquad i = 0, 1, 2, \ldots$$

(ii) $N < \infty$

$$S = \left(1 + \rho + \frac{\rho^2}{2!} + \cdots + \frac{\rho^N}{N!}\right)$$

$$\pi_i = \frac{\rho^i}{i!}\pi_0$$

$$\pi_i = \left(\frac{\rho^i}{i!} \Big/ \sum_{j=0}^{N} \frac{\rho^j}{j!}\right) \qquad i = 0, 1, \ldots N$$

$$P[\text{link saturation}] = P[N_t = N]$$

$$= \frac{(\rho^N/N!)}{\sum_{j=0}^{N}\left(\frac{\rho^j}{j!}\right)}$$

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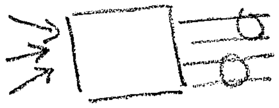
Q1 (ii)

IPP

OVERFLOW TRAFFIC MODEL



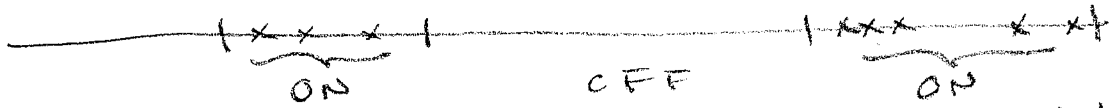
M channels (1st choice)

N channels (2nd choice)

The overflow traffic can be approximated assuming that the arrival process of the overflow traffic consists of a Poisson arrival stream which is:

ON when the first-choice link is saturated

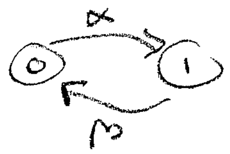
OFF when the first-choice link is not saturated



(i) ON periods : exponentially R.V. with mean $(M\mu)^{-1}$

(ii) OFF periods : are NOT exponential R.V.

If we assume that the OFF periods are exponential: The ON/OFF switching process can be represented by a 2-state Markov process $\{Y_t\}$



$1/\alpha$ = mean OFF period

$1/\beta$ = mean ON period

at equilibrium

$\pi_0 = \beta / (\alpha+\beta)$, $\pi_1 = \alpha / (\alpha+\beta)$

Therefore

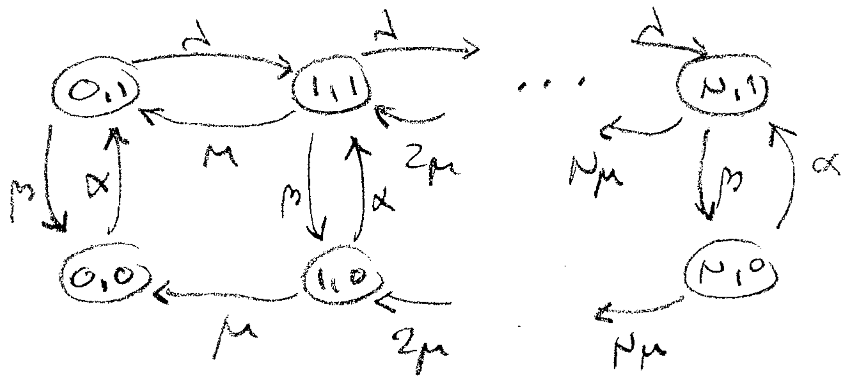
Mean overflow arrival rate $\bar{\lambda} = \frac{\alpha}{\alpha+\beta} \lambda$

Mean offered overflow traffic $\bar{\rho} = \frac{\alpha}{\alpha+\beta} \rho$

Q1 (v) Lets offer this traffic to an overflow link of size N, let N_t be the number of busy channels on this overflow link.

then the joint process $\{N_t, Y_t\}$ is a 2-dimensional B/D process called Interrupted Poisson process

State transition diagram



Q2 (a) If q_{ij} is the transition rate from i to j the equilibrium equations are:

$$\sum_i \pi_i q_{ij} = 0 \quad \text{for each state } j$$

i.e.

$$\sum_{i \neq j} \pi_i q_{ij} = -\pi_j q_{jj} = \sum_{i \neq j} \pi_j q_{ji}$$

These are the global balance equations - one for each state j. If the process is reversible the reverse transition rates $\hat{q}_{ji}$ must be the same as the corresponding forward transition rates q_{ji}

i.e.

$$\hat{q}_{ji} = q_{ji}$$

but $\hat{q}_{ji} = \frac{\pi_i}{\pi_j} q_{ij}$ (easily shown)

and so:

$$\pi_i q_{ij} = \pi_j q_{ji}$$

These are the local balance equations - one for each pair of states i, j

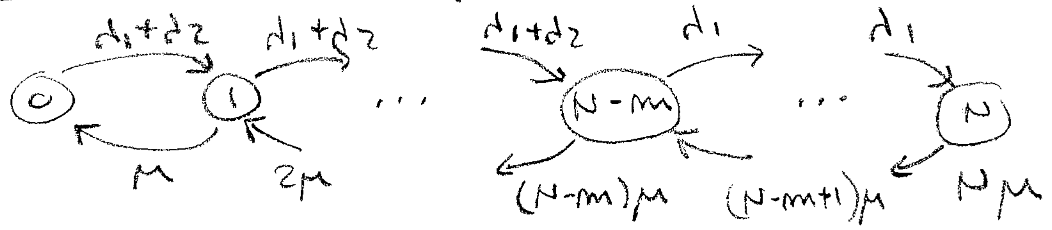
Q2 (b) The required birth/death model has

Birth coeff $\lambda_i = \lambda_1 + \lambda_2$, $i < N-m$

$= \lambda_1$, $N-m \le i < N$

Death coeff $\mu_i = i\mu$, $0 < i \le N$ $(\mu = 1/h)$

<u>state transition diagram</u>



<u>equilibrium equations</u>

$$\pi_i = \left(\frac{\lambda_{i-1}}{\mu_i}\right)\pi_{i-1} = \left(\frac{\lambda_1+\lambda_2}{i\mu}\right)\pi_{i-1}, \quad i \le N-m$$

$$= \left(\frac{\lambda_1}{i\mu}\right)\pi_{i-1}, \quad i > N-m$$

which together with

$$\sum_{i=0}^{N} \pi_i = 1$$

yield the state distribution $(\pi_0, \pi_1, \ldots, \pi_N)$

<u>Blocking probabilities</u>

for 1st choice traffic $B_1 = \pi_N$

for 2nd choice traffic $B_2 = \sum_{i=N-m}^{N} \pi_i$

Q3(a) M/G/1

Poisson arrival stream, general service time distribution and infinite capacity buffer.

Use mean value analysis

For the ith arrival to the system

let R_i = residual service time (time until first departure seen by ith arrival.

S_i = service time
W_i = waiting time
Q_i = queue length found on arrival

for the ith arrival

Taking expectations:

$$E(W_i) = E(R_i) + E\left[\sum_{1}^{Q_i} S_{i-j}\right]$$

$$= E(R_i) + E(Q_i)E(S)$$

but since Poisson arrivals see an unbiased sample of queue behaviour $E(Q_i) = E(Q_t) \Rightarrow$

$$E(W) = E(R) + E(Q)\,E(S)$$

by Little's formula

$$E(Q) = \lambda E(W)$$

$$E(W) = E(R) + \rho E(W) \qquad \rho = \lambda E(S)$$

for M/G/1 FIFO

$$E(W) = \left[\frac{E(R)}{1-\rho}\right]$$

R_t = residual service time seen by a virtual arrival at time t. Then at equilibrium, $\{R_t\}$ is a continuous time stochastic process.

assuming $\{R_t\}$ is ergodic

$$E(R_t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T R_t\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{M_T}\left(\tfrac{1}{2}S_i^2\right)$$

M_T = no. of completed services in $[0,T]$

$$E(R_t) = \lim_{T\to\infty} \underbrace{\frac{1}{2}\left(\frac{M_T}{T}\right)}_{\text{service completion rate} \,=\, \text{mean arrival rate, } \lambda}\underbrace{\left[\frac{1}{M_T}\sum_{i=1}^{M_T} S_i^2\right]}_{\text{mean square service time} \,=\, E(S^2)}$$

$$E(R_t) = \frac{1}{2}\lambda E(S^2)$$

$$E(W) = \left[\frac{\lambda E(S^2)}{2(1-\rho)}\right]$$

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Q3(b) (i) Mean service time $= \frac{700}{8}$ sec

Service Rate $= \frac{8}{700}$ sec^{-1} $= \mu$

Arrival rate $= 0.01$ sec^{-1} $= \lambda$

$\Rightarrow$ offered traffic, $\rho = \left(\frac{\lambda}{\mu}\right) = \frac{7}{8} = 0.875$ erlangs

Then for this M/M/1 system

$P[W=0] = 1-\rho = 0.125$

(ii) $P[W>600] = P[W>0]\,P[W>600 \mid W>0]$

$= \rho e^{-\mu(1-\rho)600}$

$= 0.875\, e^{-(8/700)(1/8)600}$

$= 0.37$

(iii) Giving priority to shorter files will reduce the overall mean waiting time

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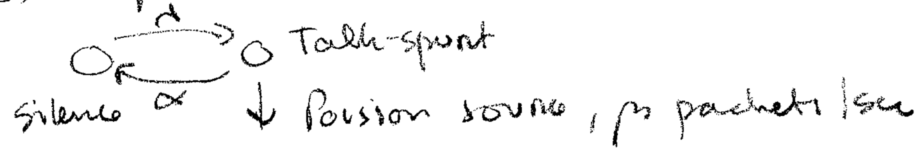
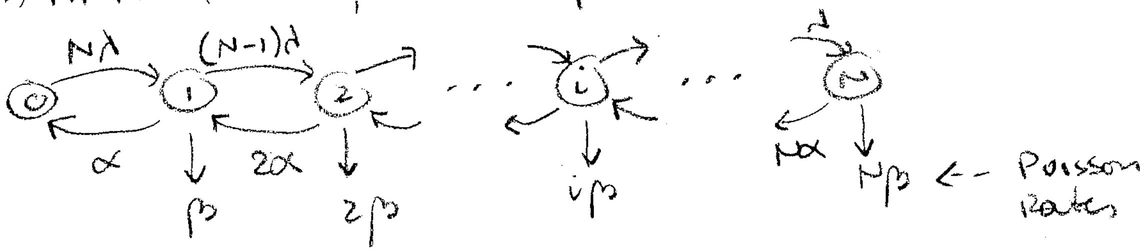
Q4 (a) m states MMPP: while in state i, $1 \le i \le m$ behaves as a Poisson process with state-dependent rate parameter λ_i. Transition between states are governed by an underlying CTMC.

<u>Poisson Model</u>

(a) Single voice source



(b) MMPP model, N multiplexed voice sources.



- multiplexer service-time distribution: exponential distribution with parameter (D)
- each source delivers an average of β cells/sec, the average number of cells/sec entering the queue is: $N\beta \left\{ \frac{\lambda}{\alpha+\lambda} \right\}$
- This must be less than the capacity of the multiplexer so:

$$N\beta \left\{ \frac{\lambda}{\alpha+\lambda} \right\} < D$$

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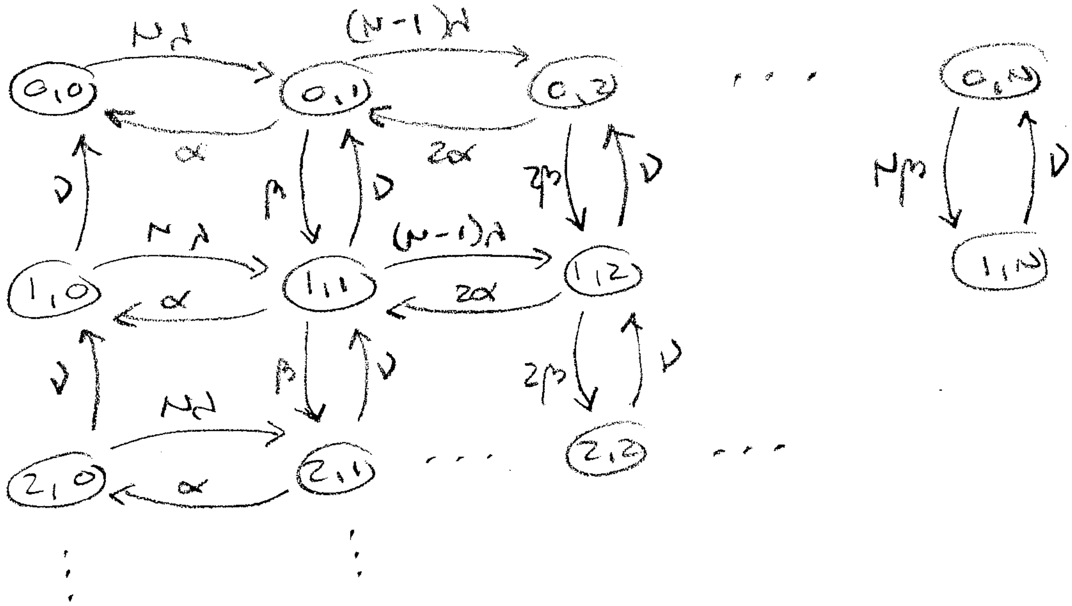
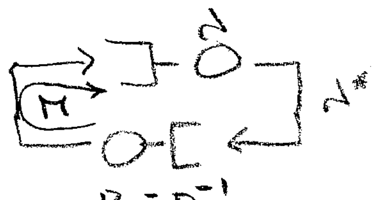
Q4 (a) State space representation of the multiplexer

define state of system

$E = \{$ queue length i, j source on $\}$



Q4 (b) Simple approximate analysis of the leaky bucket based on:



- cell generated at a Poisson rate λ
- only generated if the upper queue has at least one "occupant" or token
- The upper queue increases at an average rate $R = D^{-1}$
- M tokens circulating

Q4 (b)

- If M circulating tokens are in the "lower" queue, the upper queue is empty and cell service is blocked

$$\lambda^* = \lambda(1 - P_L)$$

P_L = probability that upper queue is empty or the lower queue is full

Either queue behaves as finite M/M/1/M =>

Probability lower queue full $= \dfrac{\rho^M (1-\rho)}{1-\rho^{M+1}}$

$$\rho = \frac{\lambda}{R} = \lambda D$$

$$\lambda^* = \lambda \left[\frac{1-\rho^M}{1-\rho^{M+1}} \right]$$

note: as M increases, throughput-load characteristic approach the ideal characteristics in which the throughput λ^* equal the load λ for $\lambda \le R$ and saturates at the maximum value of R for $\lambda \gg R$

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Q5(a) Stochastic Knapsack

- C resource units
- class k: Poisson arrivals λ_k; exponential holding time $1/\mu_k$
  hold b_k resource units
- state of the system $n = (n_1, \cdots, n_K)$
  $b = (b_1, \cdots, b_K)$

$$S = \{ n \in I^K : b \cdot n \le C \}$$

Blocking probability of class k

$$S_K = \{ n \in S : b \cdot n \le C - b_k \}$$

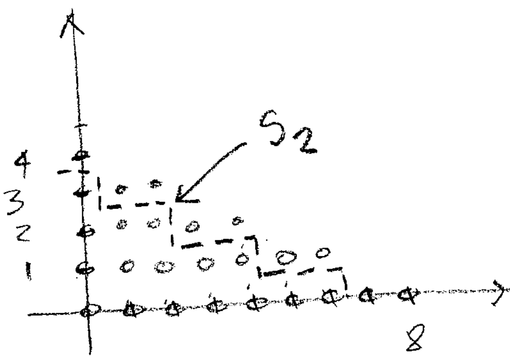
$$B_K = 1 - \sum_{n \in S_K} \pi(n)$$

$$\pi(n) = \frac{1}{G} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}$$

$$G = \sum_{n \in S} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}$$

$C = 8$; $b_1 = 1$; $b_2 = 2$.

S = all •



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Q5 (6)

(i) To find equivalent capacity function, two techniques are employed: (i) fluid flow models and (ii) stationary approximation. The two techniques give reasonably correct equivalent capacity function in different regions.

- Fluid flow model is a good model when the impact of individual connections is critical.
- Stationary approximation works well when the effect of statistical multiplexing is of significance.
- Both techniques are typically exclusive and conservative

(ii) Equivalent capacity

$$C_L = (m + K\sigma) R_p$$

$m R_p$ = mean

σR_p = standard deviation

$$K = K(QoS)$$

$$m = Np \quad \text{(on-off source model)}$$

$$\sigma^2 = Np(1-p) = m(1-p)$$

$$C = \frac{C_L}{R_p} \rightarrow C = m + K\sigma$$

$$C = Np + K\sqrt{Np(1-p)}, \quad K(QoS) \sim P_L$$

(i) $$P_L = \sum_{i=J_0}^{N} \frac{(i-c)\pi_i}{m}$$

(ii) $$\varepsilon = \sum_{i=J_0}^{N} \pi_i$$

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Q5 (b) (ii) Effect of large number of sources multiplexed

$N \gg 1$, $p \ll 1$

$\pi_i = \binom{N}{i} p^i (1-p)^{N-i}$ (binomial)

Is approximated quite closely by the normal distribution ($m = Np$, $\sigma^2 = Np(1-p)$)

$$P_L = \frac{1}{m} \int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} (x-c)\, dx$$

$$\varepsilon = \int_{J_0}^{\infty} \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\, dx$$

if $(c-m) > 3\sqrt{2}\,\sigma$

$$\varepsilon = \frac{\sigma e^{-(c-m)^2/2\sigma^2}}{\sqrt{2\pi}\,(c-m)} \; ; \quad P_L = \frac{1-p}{c-m}\varepsilon$$

$$\ln(\sqrt{2\pi}\,\varepsilon) = \ln\left(\frac{\sigma}{c-m}\right) - \frac{(c-m)^2}{2\sigma^2}$$

$$c_{LS} = m R_p + \sigma \underbrace{\sqrt{-\ln(2\pi) - 2\ln\varepsilon}}_{K}\, R_p$$

Q5 (a) (iii) Effect of access buffer

$G(x) \sim A_N \rho^N e^{-\beta_R x / R_p}$

(probability buffer occupancy $> x$)

$R = (1-\rho)\left(1 + \frac{\alpha}{\beta}\right) \Big/ \left(1 - \frac{C_L}{N R_p}\right)$

$\rho = \dfrac{N p R_p}{C_L}$

if $\rho \sim 1$ $\quad A_N \rho^N \sim 1$

$P_L = e^{-\beta_R x / R_p}$

$\beta_R x / R_p = -\ln P_L$

$\dfrac{C_L}{R_p N} = \dfrac{1-k}{2} + \sqrt{\left(\dfrac{1-k}{2}\right)^2 + kp}$

$C_{LF} = R_p N \left(\dfrac{1-k}{2}\right) + R_p N \sqrt{\left(\dfrac{1-k}{2}\right)^2 + kp}$

$C_L = \min\left[C_{LS}, C_{LF}\right]$