1.a Consider first the signal :

$$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is :

$$\begin{aligned} g(t) &= y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^{n} c_{n-i+1} \delta((i-1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1} \delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$. Since,

$$s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) \star \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is :

$$S(f) = P(f) \sum_{k=1}^{n} c_k e^{-j2\pi f k T_c}$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is :

$$\begin{aligned} H(f) &= S^*(f) e^{-j2\pi f T} = S^*(f) e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{k=1}^{n} c_k e^{j2\pi f k T_c} e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f (i-1) T_c} \\ &= P^*(f) \mathcal{F}[g(t)] \end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is

$$\begin{aligned} \int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c) dt \\ &= T_c \sum_{k=1}^{n} c_k^2 \end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

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MODEL ANSWERs and MARKING SCHEME

First Examiner: Gurcan, M.K. Paper Code : E4.04, SC6, ISE4.9

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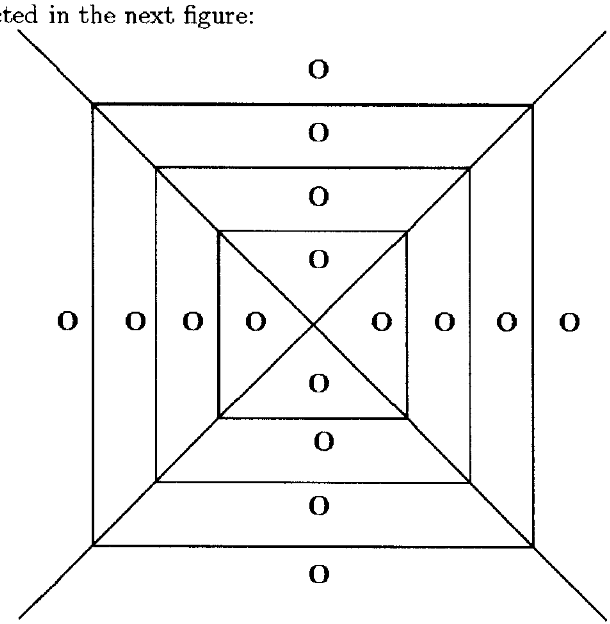
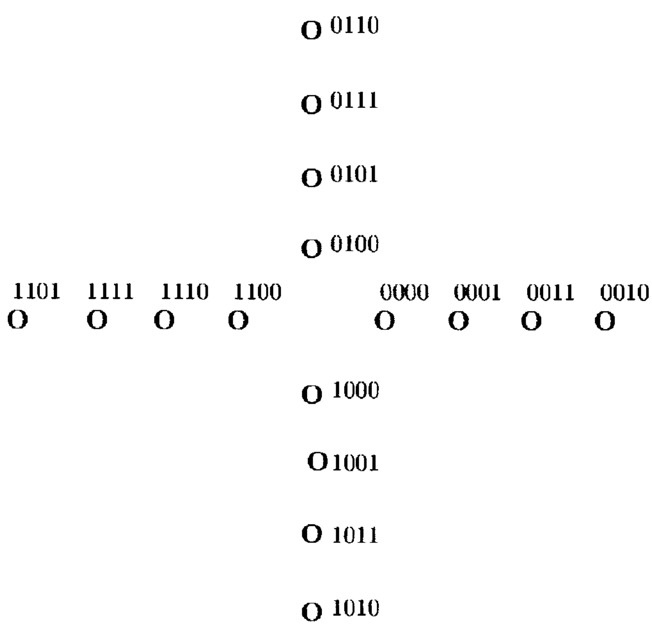
1.b The optimum decision boundary of a point is determined by the perpedicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:



One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure.



1.c

1) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, this is not the case for the 8-QAM constellation of Figure This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$\mathrm{B} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad \mathrm{C} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2) Since each symbol conveys 3 bits of information, the resulted symbol rate is

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

3) The probability of error for an M-ary PSK signal is

$$P_M = 2Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} \sin\frac{\pi}{M}\right]$$

whereas the probability of error for an M-ary QAM signal is upper bounded by

$$P_M = 4Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

Since, the probability of error is dominated by the argument of the Q function, the two signals will achieve the same probability of error if

$$\sqrt{2\mathrm{SNR_{PSK}}} \sin\frac{\pi}{M} = \sqrt{\frac{3\mathrm{SNR_{QAM}}}{M-1}}$$

With $M = 8$ we obtain

$$\sqrt{2\mathrm{SNR_{PSK}}} \sin\frac{\pi}{8} = \sqrt{\frac{3\mathrm{SNR_{QAM}}}{7}} \Longrightarrow \frac{\mathrm{SNR_{PSK}}}{\mathrm{SNR_{QAM}}} = \frac{3}{7 \times 2 \times 0.3827^2} = 1.4627$$

4) Assuming that the magnitude of the signal points is detected correctly, then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5^o. In the case of the 8-QAM constellation an error will be made if the magnitude phase error exceeds 45^o. Hence, the QAM constellation is more immune to phase errors.

1.d

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant n is given by

$$\theta(n;\mathbf{a}) = \frac{\pi}{2}\sum_{k=0}^{k} a_k + \theta_0$$

where θ_0 is the initial phase and a_k is ± 1 depending on the input bit at the time instant k. The following table shows $\theta(n;\mathbf{a})$ for two different values of θ_0 $(0, \pi)$, and the four input pairs of data: $\{00, 01, 10, 11\}$.

θ_0	b_0	b_1	a_0	a_1	$\theta(n;\mathbf{a})$
0	0	0	-1	-1	$-\pi$
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

2.a

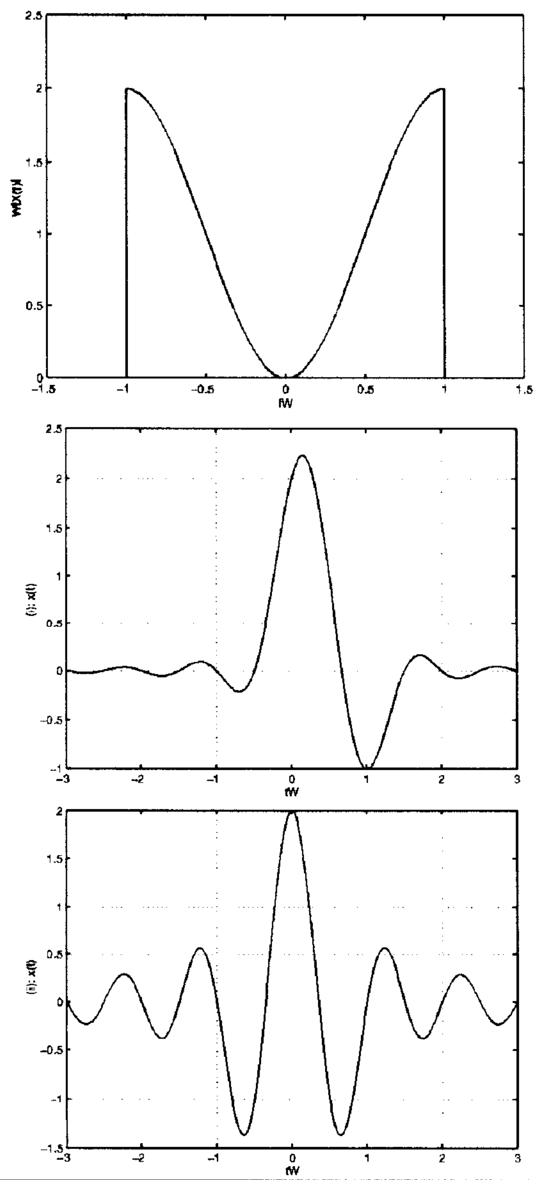
$x_{-1} = -1,\ x_0 = 2, x_1 = -1$, otherwise $x_n = 0$. Then :

$$x(t) = 2\frac{\sin(2\pi Wt)}{2\pi Wt} - \frac{\sin(2\pi W(t+1/2W))}{2\pi W(t+1/2W)} - \frac{\sin(2\pi W(t-1/2W))}{2\pi W(t-1/2W)}$$

and :

$$X(f) = \frac{1}{2W}\left|2 - e^{-j\pi f/W} - e^{+j\pi f/W}\right| = \frac{1}{2W}\left|2 - 2\cos\frac{\pi f}{W}\right| = \frac{1}{W}\left|1 - \cos\frac{\pi f}{W}\right|,\ |f| \leq W$$

The plot of $|X(f)|$ is given in the following figure :



2.a Cont.

$$B_n = 2I_n - I_{n-1} - I_{n+1}$$

where $I_m = \pm 1$. Hence :

$$P(B_n = 0) = 1/4$$
$$P(B_n = -2) = 1/4$$
$$P(B_n = 2) = 1/4$$
$$P(B_n = -4) = 1/8$$
$$P(B_n = 4) = 1/8$$

2.b

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by :

$$1600(1+\beta) = 2000$$

which yields $\beta = 0.25$. Since β is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain :

$$R = \frac{9600}{4} = 2400$$

and :

$$1200(1+\beta) = 2000$$

or $\beta = 2/3$, which satisfies the required conditions. The probability of error for an M-QAM constellation is given by :

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where :

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

With $P_M = 10^{-6}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore using the last equation and the table of values for the $Q(\cdot)$ function, we find that the average transmitted energy is :

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is :

$$\int_{-\infty}^{\infty} g_T^2(t)dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{-\infty}^{\infty} X_{rc}(f)df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T. Since $T = \frac{1}{2400}$, the average transmitted power is :

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

If the points of the 16-QAM constellation are evenly spaced with minimum distance between them equal to d, then there are four points with coordinates $(\pm\frac{d}{2}, \pm\frac{d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{3d}{2})$, and eight points with coordinates $(\pm\frac{3d}{2}, \pm\frac{d}{2})$, or $(\pm\frac{d}{2}, \pm\frac{3d}{2})$. Thus, the average transmitted power is :

$$P_{av} = \frac{1}{2 \times 16}\sum_{i=1}^{16}(A_{mc}^2 + A_{ms}^2) = \frac{1}{32}\left[4 \times \frac{d^2}{2} + 4 \times \frac{9d^2}{2} + 8 \times \frac{10d^2}{4}\right] = \frac{5}{4}d^2$$

Since $P_{av} = 592.8 \times 10^{-7}$, we obtain

$$d = \sqrt{4\frac{P_{av}}{5}} = 0.0069$$

2.c Let $X(f)$ be such that

$$\mathrm{Re}[X(f)] = \begin{cases} T\Pi(fT) + U(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{Im}[X(f)] = \begin{cases} V(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases}$$

with $U(f)$ even with respect to 0 and odd with respect to $f = \frac{1}{2T}$ Since $x(t)$ is real, $V(f)$ is odd with respect to 0 and by assumption it is even with respect to $f = \frac{1}{2T}$. Then,

$$\begin{aligned} x(t) &= \mathcal{F}^{-1}[X(f)] \\ &= \int_{-\frac{1}{T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{\frac{1}{2T}}^{\frac{1}{T}} X(f)e^{j2\pi ft}df \\ &= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} Te^{j2\pi ft}df + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df \\ &= \mathrm{sinc}(t/T) + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df \end{aligned}$$

Consider first the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$. Clearly,

$$\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df = \int_{-\frac{1}{T}}^{0} U(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$$

and by using the change of variables $f' = f + \frac{1}{2T}$ and $f' = f - \frac{1}{2T}$ for the two integrals on the right hand side respectively, we obtain

$$\begin{aligned} &\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df \\ &= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\ &\stackrel{a}{=} \left(e^{j\frac{\pi}{T}t} - e^{-j\frac{\pi}{T}t}\right) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\ &= 2j\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \end{aligned}$$

where for step (a) we used the odd symmetry of $U(f')$ with respect to $f' = \frac{1}{2T}$, that is

$$U(f' - \frac{1}{2T}) = -U(f' + \frac{1}{2T})$$

For the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df$ we have

$$\begin{aligned} &\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\ &= \int_{-\frac{1}{T}}^{0} V(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\ &= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T})e^{j2\pi f't}df' \end{aligned}$$

However, $V(f)$ is odd with respect to 0 and since $V(f' + \frac{1}{2T})$ and $V(f' - \frac{1}{2T})$ are even, the translated spectra satisfy

$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T})e^{j2\pi f't}df' = -\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T})e^{j2\pi f't}df'$$

Hence,

$$\begin{aligned} x(t) = \; &\mathrm{sinc}(t/T) + 2j\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\ &-2\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \end{aligned}$$

and therefore,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Thus, the signal $x(t)$ satisfies the Nyquist criterion.

2.d

If $\{c_n\}$ denote the coefficients of the zero-force equalizer and $\{q_m\}$ is the sequence of the equalizer's output samples, then :

$$q_m = \sum_{n=-1}^{1} c_n x_{m-n}$$

where $\{x_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_b$, we obtain the system :

$$\begin{pmatrix} \mathcal{E}_b & 0.9\mathcal{E}_b & 0.1\mathcal{E}_b \\ 0.9\mathcal{E}_b & \mathcal{E}_b & 0.9\mathcal{E}_b \\ 0.1\mathcal{E}_b & 0.9\mathcal{E}_b & \mathcal{E}_b \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}_b \\ \mathcal{E}_b \end{pmatrix}$$

The solution to the system is :

$$\begin{pmatrix} c_{-1} & c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0.2137 & -0.3846 & 1.3248 \end{pmatrix}$$

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MODEL ANSWERs and MARKING SCHEME

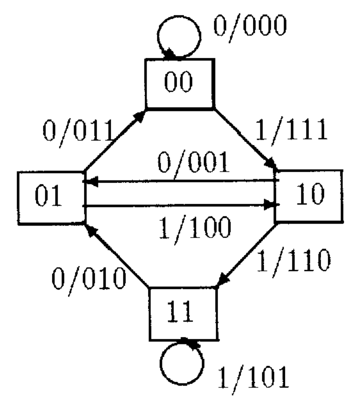
First Examiner: **Gurcan, M.K.** **Paper Code : E4.04, SC6, ISE4.9**

Second Examiner: Constantinides, A.G. **Question 3 Page 10** ***out of 23***

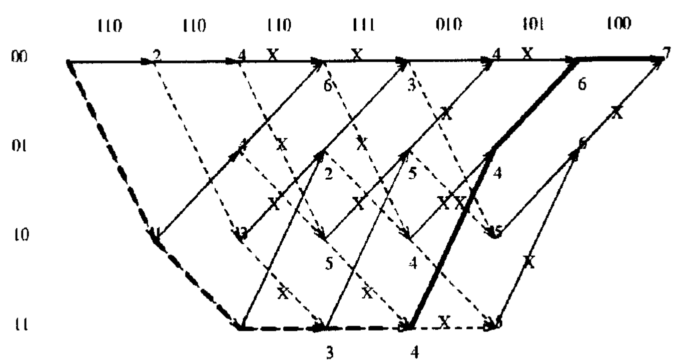
Question labels in left margin **Marks allocations in right margin**

3.a

The state transition diagram for this code is depicted in the next figure.



The following figure shows 7 frames of the trellis diagram used by the Viterbi decoder. It is assumed that the input sequence is padded by two zeros, so that the actual length of the information sequence is 5. The numbers on the nodes indicate the Hamming distance of the survivor paths. The deleted branches have been marked with an X. In the case of a tie we deleted the upper branch. The survivor path at the end of the decoding is denoted by a thick line.



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First Examiner: **Gurcan, M.K.** **Paper Code :** **E4.04, SC6, ISE4.9**

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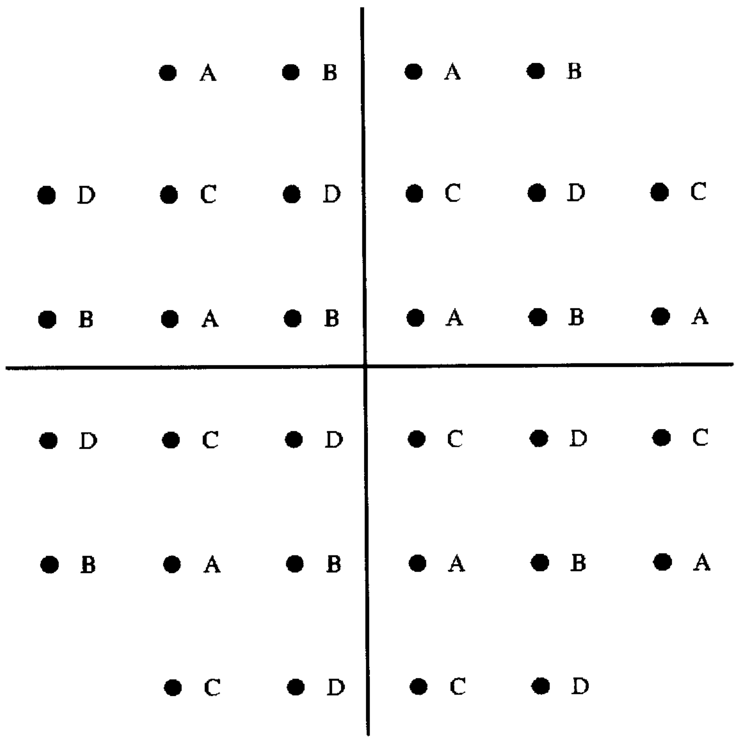
3.b

There are 4 subsets corresponding to the four possible outputs from the rate 1/2 convolutional encoder. Each subset has eight signal points, one for each of the 3-tuples from the uncoded bits. If we denote the sets as A,B,C,D, the set partitioning is as follows :



The minimum distance between adjacent points in the same subset is doubled.

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MODEL ANSWERs and MARKING SCHEME

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Second Examiner: Constantinides, A.G. Question 3 Page 12 *out of 23*

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3-c

Signal constellation points are

Signal s_i	$s_{i,1}$	$s_{i,2}$
s_0	0.9239	0.3827
s_1	0.3827	0.9239
s_2	-0.3827	0.9239
s_3	-0.9239	0.3827
s_4	-0.9239	-0.3827
s_5	-0.3827	-0.9239
s_6	0.3827	-0.9239
s_7	0.9239	-0.3827

Received signal samples are

Received signal r_i	$r_{i,1}$	$r_{i,2}$
r_1	0.910	0.390
r_2	0.390	-0.940
r_3	0.340	0.940
r_4	-0.390	0.950
r_5	-0.430	0.900
r_6	0.870	-0.370
r_7	-0.370	-0.890
r_8	0.880	-0.350
r_9	0.930	0.350
r_{10}	0.420	0.910
r_{11}	-0.400	0.970

3.c cont.

As shown in the one stage trellis diagram shown in Figure 1 we have two branch in between two states between two consecutive time slots. We calculate the Euclidean distance between the received signal samples and also each branch in the trellis.

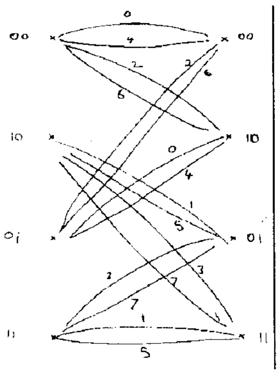

Figure 1

For each parallel branch we find the branch with the minimum distance to the received signal. We use the Euclidean distance as the branch metric for the Viterbi algorithm. In tables 1 to 4 we calculate the Euclidean distances for the branch metrices.

Time slot k	$\lvert r_k - s_0\rvert^2$	$\lvert r_k - s_4\rvert^2$	$\min(\lvert r_k - s_0\rvert, \lvert r_k - s_4\rvert)$ (branch metric)	Selected signal	Input data causing the transition between states 00 and 00	Input data causing the transition between states 01 and 10
1	0.00025	3.96015	0.016	s_0	00	01
2	2.03452	2.03688	1.426	s_0	00	01
3	0.65152	3.34688	0.807	s_0	00	01
4	2.04813	2.06107	1.431	s_0	00	01
5	2.10061	1.88919	1.374	s_4	10	11
6	0.56944	3.21816	0.755	s_0	00	01
7	3.29385	0.56415	0.751	s_4	10	11
8	0.53875	3.25505	0.734	s_0	00	01
9	0.00111	3.97369	0.033	s_0	00	01
10	0.53196	3.47704	0.729	s_0	00	01
11	2.09760	2.10420	1.448	s_0	00	01

Table 1

3.c cont.

Time slot k	$\lvert r_k - s_1\rvert^2$	$\lvert r_k - s_5\rvert^2$	$\min(\lvert r_k - s_1\rvert, \lvert r_k - s_5\rvert)$ (branch metric)	Selected signal	Input data causing the transition between states 10 and 01	Input data causing the transition between states 11 and 11
1	0.56309	3.39731	0.750	s_1	00	01
2	3.47410	0.59730	0.773	s_5	10	11
3	0.00208	3.99632	0.046	s_1	00	01
4	0.59772	3.51148	0.773	s_1	00	01
5	0.66102	3.32878	0.813	s_1	00	01
6	1.91160	1.87600	1.370	s_5	10	11
7	3.85669	0.00131	0.036	s_5	10	11
8	1.87009	1.92371	1.368	s_1	00	01
9	0.62889	3.34591	0.793	s_1	00	01
10	0.00159	4.00741	0.040	s_1	00	01
11	0.61472	3.58708	0.784	s_1	00	01

Table 2

Time slot k	$\lvert r_k - s_2\rvert^2$	$\lvert r_k - s_6\rvert^2$	$\min(\lvert r_k - s_2\rvert, \lvert r_k - s_6\rvert)$ (branch metric)	Selected signal	Input data causing the transition between states 00 and 10	Input data causing the transition between states 01 and 00
1	1.95606	2.00434	1.399	s_2	01	00
2	4.07109	0.00031	0.018	s_6	11	01
3	0.52253	3.47587	0.723	s_2	01	00
4	0.00074	4.10846	0.027	s_2	01	00
5	0.00281	3.98699	0.053	s_2	01	00
6	3.24334	0.54426	0.738	s_6	11	10
7	3.29032	0.56768	0.753	s_6	11	10
8	3.21714	0.57666	0.759	s_6	11	10
9	2.05248	1.92232	1.386	s_6	11	10
10	0.64449	3.36451	0.803	s_2	01	00
11	0.00243	4.19937	0.049	s_2	01	00

Table 3

3.c cont.

Time slot k	$\|r_k - s_3\|^2$	$\|r_k - s_7\|^2$	$\min(\|r_k - s_3\|, \|r_k - s_7\|)$ (branch metric)	Selected signal	Input data causing the transition between states 10 and 11	Input data causing the transition between states 11 and 01
1	3.36317	0.59723	0.77281	s_7	11	10
2	3.47577	0.59563	0.77177	s_7	11	10
3	1.90799	2.09041	1.38130	s_3	01	00
4	0.60688	3.50232	0.77902	s_3	01	00
5	0.51153	3.47827	0.71522	s_3	01	00
6	3.78454	0.00306	0.05535	s_7	11	10
7	1.92651	1.93149	1.38799	s_3	01	00
8	3.79081	0.00299	0.05471	s_7	11	10
9	3.43794	0.53686	0.73271	s_7	11	10
10	2.08407	1.92493	1.38742	s_7	11	10
11	0.61939	3.58241	0.78701	s_3	01	00

Table 4

Using columns 5,6 and 7 of tables 1- 4 we produce the reduced trellis diagram as shown in Figure 2. The notation used for each branch identifies the signal constellation point with minimum distance to the received signal. Underneath of each signal constellation point we provide the input data causing the transition from one state to the next.

In figure 3. we produce the branch metrices for each branch.

In figure 4 we provide the path metrices that are produced using the Viterbi algorithm and the branch metrices provided in figure 3.

In figure 5 we uses the path metric at the termination point and trace back the states with minimum path metrices. We provide the signal numbers and also the input data causing the transitions between states. From Figure 5 we identified that The received signal samples correspond to the transmitted data sequence

00 11 00 00 01 11 11 10 01 00 00.

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Question labels in left margin **Marks allocations in right margin**

3.c cont.

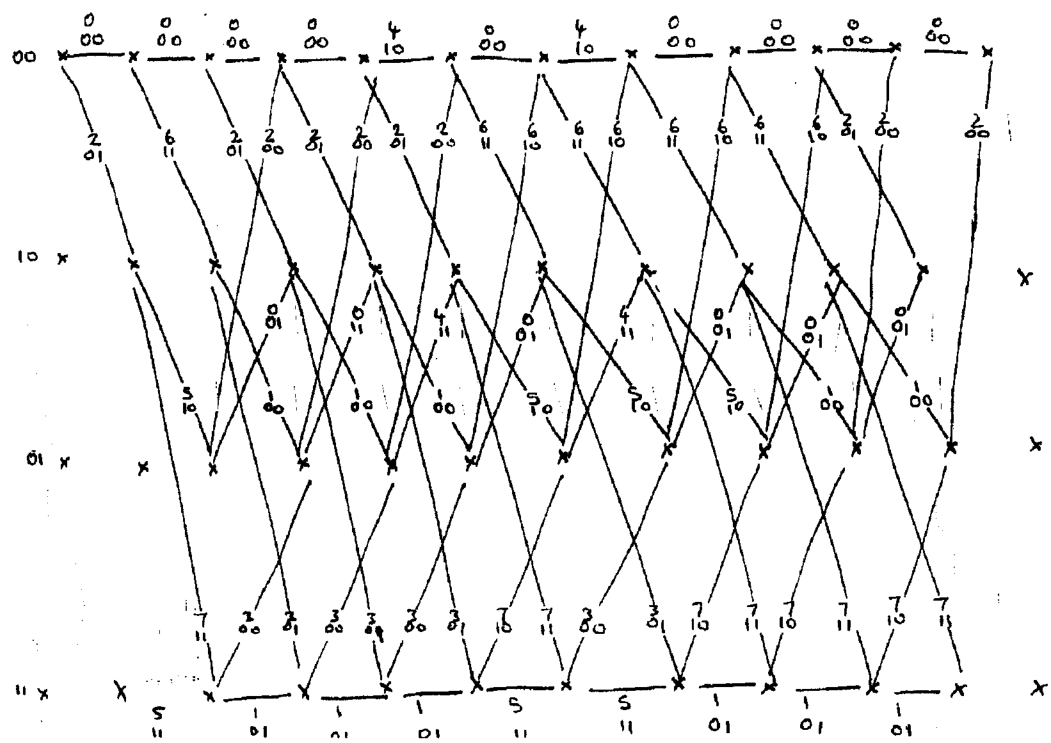

Figure 2

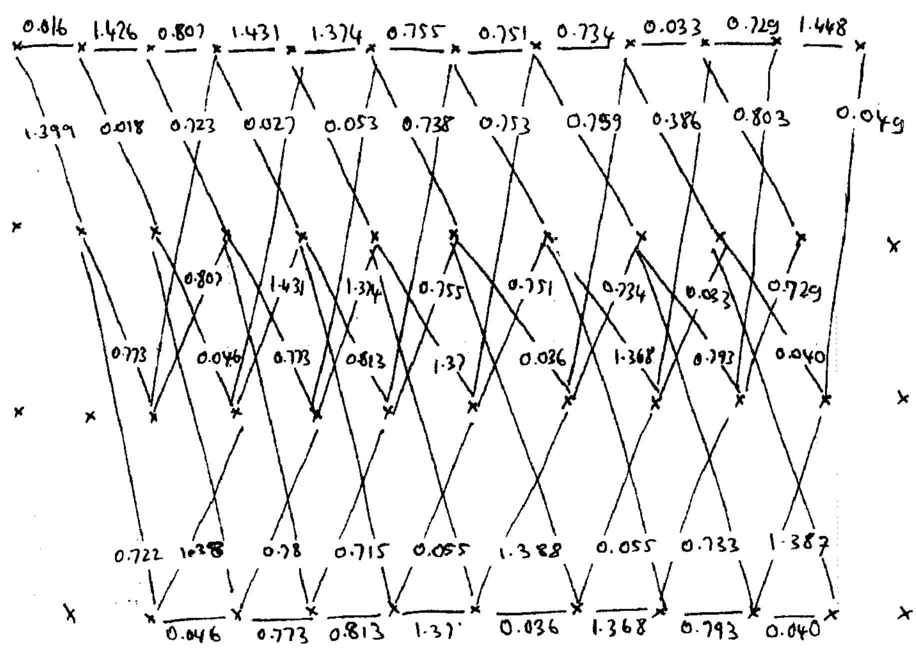

Figure 3

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Question labels in left margin **Marks allocations in right margin**

3.c cont.

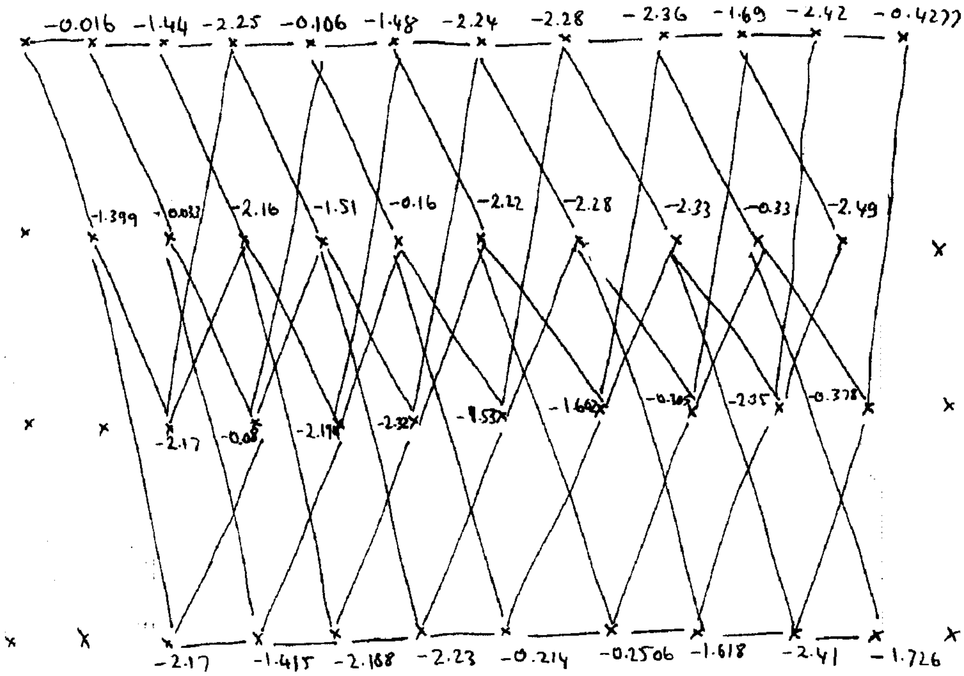

Figure 4

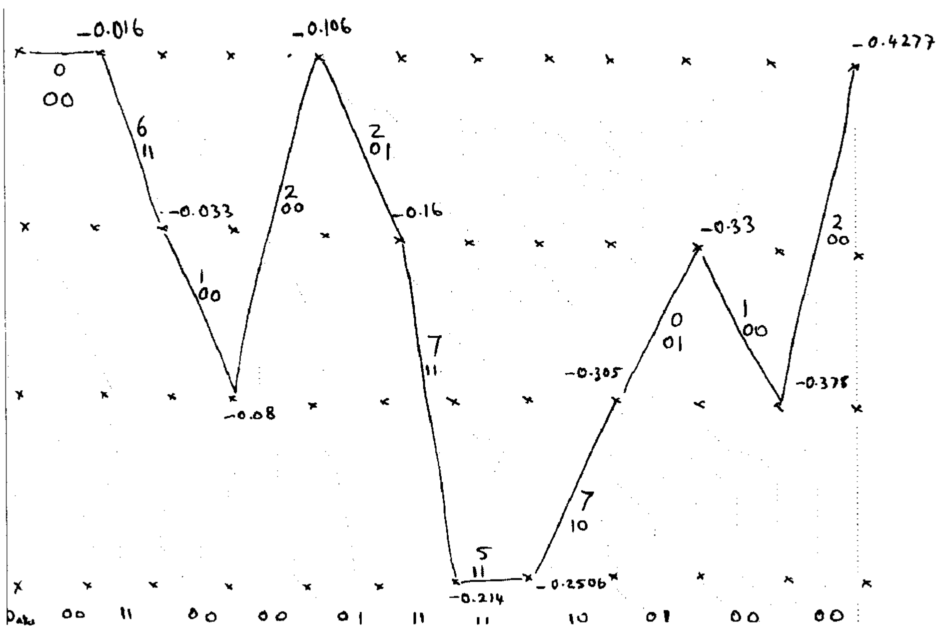

Figure 5

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MODEL ANSWERs and MARKING SCHEME	
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4.a

$$SNR_{in}(dB) = 30\exp\left(-\frac{1.5\,f}{1.1\quad 10^{6}}\right)$$

$N = 256$

$f_1 = 25\ kHz$

$\Delta f = 4.3125\quad kHz$

$f_{256} = 1124.7\quad kHz$

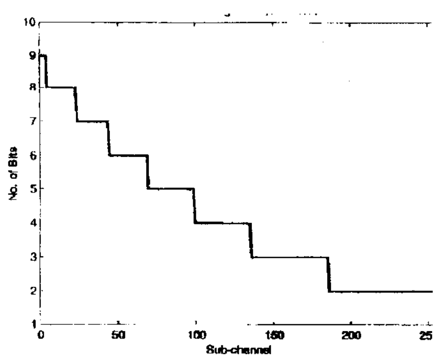
$T = 250\quad \mu s$

$$T_s = \frac{T}{256} = \frac{250}{256}\quad \mu s$$

$$r_s = \frac{1}{T_s}$$

Centre Frequency (kHz)	SNR$_{in}$ (dB)
$f_1 = 25$	28.9945
$f_2 = 29.3125$	28.8245
$f_3 = 33.6250$	28.6555
...	...
$f_{256} = 1124.7$	6.4723

$P_{e,s} = 0.05$



$$\text{Total bit rate} = 1062\ \frac{1}{T} = \frac{1062}{250\quad 10^{-6}} = 4\,248\,000\quad bits/s$$

4.b

If the sample rate $\frac{1}{T_s} = \tilde{N} \cdot \Delta f = W$, does not alter with the insertion of the cyclic prefix (which indeed is the case in most multicarrier systems), then the bandwidth requirements for the system remain the same. However, keeping the same sample rate means that the block length is increased by a factor of $\frac{v}{N}$, and the effective throughput is reduced to $\frac{1}{1+\frac{v}{N}} = \frac{N}{N+v}$ of the previous one. This is usually compensated by the elimination of ISI, which allows the use of higher order alphabets in each one of the subcarriers.
If the sample rate is increased by a factor of $\left(\frac{N}{N+v}\right)^{-1}$, so that the block length after the insertion of the cyclic prefix will be the same as before, then the bandwidth requirements for the system are increased by the same factor : $W' = W\frac{N+v}{N}$. However, this second case is rarely used in practice.

If the real and imaginary parts of the information sequence $\{X_k\}$ have the same average energy : $E\left[Re(X_k)\right]^2 = E\left[Im(X_k)\right]^2$, then it is straightforward to prove that the time-domain samples $\{x_n\}$, that are the output of the IDFT, have the same average energy:

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (Re(X_k) + j \cdot Im(X_k)) \exp(j2\pi nk/N), \quad n = 0, 1, ..., N-1$$

and :

$$E\left[x_n^2\right] = \epsilon$$

for all $n = 0, 1, ...N-1$. Hence, the energy of the cyclic-prefixed block, will be increased from $N\epsilon$ to $(N+v)\,\epsilon$. However, the power requirements will remain the same, since the duration of the prefixed block is also increased from NT_s to $(N+v)\,T_s$.

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MODEL ANSWERs and MARKING SCHEME

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4.c

- **The MAP Algorithm calculates three different likelihood functions**

$\alpha_{k-1}(s')$ • **is the pdf for the trellis being in state s' at time k -1 and the received channel sequence up to this point is $\tilde{y}_{j<k}$**

$\beta_k(s)$ • **is the pdf that the future received channel sequence will be $\tilde{y}_{j>k}$ given that the trellis is in state s at time k.**

$\gamma_k(s',s)$ • **is the pdf that given the trellis was in state s' at time k -1 , it moves to state s at time k and the received channel sequence for this transition is $\tilde{y}_k$**

Derivations of Likelihood functions for MAP

- **the pdf for the received channel sequence $\tilde{y}_{j<k}$ up to time k**

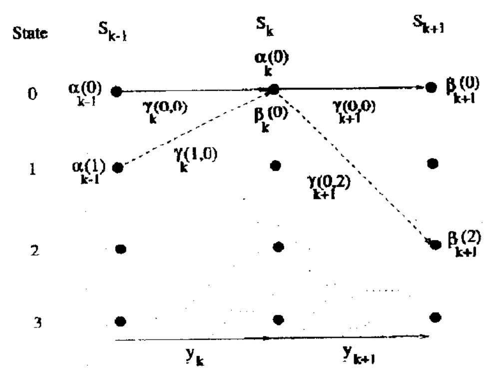
$$\alpha_{k-1}(s') = f(\tilde{y}_{j<k},\ \ S_{k-1} = s')$$

- **The pdf the future received channel sequence will be $\tilde{y}_{j>k}$ given that the current state is s**

$$\beta_k(s) = f(\tilde{y}_{j>k} \mid \ S_k = s)$$

- **The pdf given that the trellis moves to state s from state s'**

$$\begin{aligned}\gamma_k(s',\ s) &= f(\tilde{y}_k, S_k = s \mid \ \ S_{k-1} = s') \\ &= f(\tilde{y}_k \mid S_k = s,\ S_{k-1} = s')\, f(S_k = s \ \ \mid S_{k-1} = s') \\ &= f(\tilde{y}_k \mid S_k = s,\ S_{k-1} = s')\, P(u_{k,1})\end{aligned}$$



4.c

•The pdf given that the trellis moves to state s from state s'

- **In general form**

$$\gamma_k(s',\ s)=f(\tilde{y}_k \mid S_k=s,\ S_{k-1}=s')P(u_{k,1})$$

- **For rate 1/3 and first decoder for Turbo code**

$$\gamma_{k,1}(s',\ s)=f(y_{ks}\mid u_{k,1})f(y_{kp1}\mid u_{k,2})P(u_{k,1})$$
$$=f(y_{k,1}\mid u_{k,1})f(y_{k,2}\mid u_{k,2})P(u_{k,1})$$

- **For the second decoder**

$$\gamma_{k,2}(s',\ s)=f(y_{ks}\mid u_{k,1})f(y_{kp2}\mid u_{k,3})P(u_{k,1})$$
$$=f(y_{k,1}\mid u_{k,1})f(y_{k,3}\mid u_{k,3})P(u_{k,1})$$

- **Where the conditional pdf is**

$$f(y_{k,i}|u_{k,i})-\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(y_{k,i}-u_{k,i})^2}{N_0}\right)$$

•Simplification of the pdf for the Log likelihood function

- **The pdf $f(s',s,\tilde{y})$ that the encoder trellis took the transition from state $S_{k-1}=s'$ to state $S_k=s$ and the received sequence is $\tilde{y}$ can be split into the product of three terms $\alpha_{k-1}(s')$, $\gamma_k(s',s)$ and $\beta_k(s)$**

$$L(u_k|\tilde{y})=\ln\frac{\sum_{(s',s)\Rightarrow u_k=+1} f(S_{k-1}=s',S_k=s,\tilde{y})}{\sum_{(s',s)\Rightarrow u_k=-1} f(S_{k-1}=s',S_k=s,\tilde{y})}$$

$$=\ln\frac{\sum_{(s',s)\Rightarrow u_k=+1}\alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}{\sum_{(s',s)\Rightarrow u_k=-1}\alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}$$

$$=\ln\frac{\sum_{(s',s)\Rightarrow u_k=+1} f(\tilde{y}\mid u_{k,1}=+1)\ P(u_{k,1}=+1)}{\sum_{(s',s)\Rightarrow u_k=-1} f(\tilde{y}\mid u_{k,1}=+1)\ P(u_{k,1}=-1)}$$

•Log Likelihood Ratio Calculations Require

$$L(u_k|\tilde{y})=\ln\frac{\sum_{(s',s)\Rightarrow u_k=+1}\alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}{\sum_{(s',s)\Rightarrow u_k=-1}\alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}$$

- **The MAP algorithm finds $\alpha_{k-1}(s')$, and $\beta_k(s)$ for all states s throughout the trellis, i.e., for k = 0, 1. ..N -1, and for all possible transitions $\gamma_k(s',s)$ from state $S_{k-1}=s'$ to state $S_k=s$ again for k = 0, 1. ..N -1.**

4.c

- **Recursive calculation of the pdf for the received channel sequence $\tilde{y}_{j<k}$ up to time k**
- **The log likelihood Ratio calculations require $\alpha_k(s)$,**
- **From the definition of $\alpha_{k-1}(s')$, we can write**

$$\alpha_k(s) = f(\tilde{y}_{j<k+1},\ S_k = s)$$
$$= f(\tilde{y}_{j<k},\ \tilde{y}_k,\ s)$$
$$= \sum_{all\ s'} f(\tilde{y}_{j<k},\ \tilde{y}_k,\ s',\ s)$$

- **where in the last line we split the pdf $f(\tilde{y}_{j<k+1},\ s)$ into the sum of joint pdfs $f(\tilde{y}_{j<k+1},\ s',\ s)$ over an possible previous states .**

- **Recursive calculation of the pdf for the received channel sequence $\tilde{y}_{j<k}$ up to time k**

- **Using Bayes' rule and the assumption that the channel is memoryless again**

$$\alpha_k(s) = \sum_{all\ s'} f(\tilde{y}_{j<k},\ \tilde{y}_k,\ s',\ s)$$
$$= \sum_{all\ s'} f(\{\tilde{y}_k, s\} \mid \{\tilde{y}_{j<k},\ s'\})\ f(s', \tilde{y}_{j<k})$$
$$= \sum_{all\ s'} f(\{\tilde{y}_k, s\} \mid s')\ f(s', \tilde{y}_{j<k})$$
$$= \sum_{all\ s'} \gamma(s', s)\ \alpha_{k-1}(s')$$

- **Recursive calculation of the pdf that the future received channel sequence $\tilde{y}_{j>k}$ given that the current state is s**
- **Using Bayes' rule and the assumption that the channel is memoryless again**

$$\beta_{k-1}(s') = f(\tilde{y}_{j>k-1} \mid s')$$
$$= \sum_{all\ s} \beta_k(s)\ \gamma_k(s', s)$$

- **Calculation of the pdf given that the trellis moves from state s' to the state s**
- **Using the definition of $\gamma_k(s', s)$ and the derivation from Bayes' rule we have**

$$\gamma_k(s', s) = f(\{\tilde{y}_k,\ s\} \mid s')$$
$$= f(\tilde{y}_k \mid \{s,\ s'\})\ f(s \mid s')$$
$$= f(\tilde{y}_k \mid u_k)\ f(u_k) = f(\tilde{y}_k \mid u_k)\ P(u_k)$$

4.c

- **Summary of MAP Algorithm**
- **From the previous description, we see that the MAP decoding of a received sequence $\tilde{y}_k$ to give the a-posteriori LLR $L(u_k|\tilde{y})$ can be carried out as follows.**

1. **The a-priori LLRs $L(u_k)$ (which are provided in an iterative turbo decoder by the other component decoder are used to calculate $\gamma_k(s',s)$ using**

$$\gamma_k(s',s) = f(\tilde{y}_k \mid \tilde{u}_k)\ P(u_{k,1})$$

Where

$$P(u_{k,1} = \pm 1) = \frac{\exp(-L(u_k)/2)}{1+\exp(-L(u_k))}\ \exp\left(u_k \frac{L(u_k)}{2}\right)$$

And

$$f(\tilde{y}_k \mid \tilde{u}_k) = \prod_{l=1}^{n} \frac{1}{\sqrt{\pi}\ N_0} \exp\left(-\frac{1}{N_0}(y_{kl} - u_{kl})^2\right)$$

2. **The forward recursion equation**

$\alpha_k(s) = \sum_{all\ s'} \gamma(s',s)\ \ \alpha_{k-1}(s')$ **is used to calculate** $\alpha_k(s)$

3. **The backward recursion equation**

$\beta_{k-1}(s') = \sum_{all\ s} \beta_k(s)\ \ \gamma_k(s',s)$ **is used to calculate** $\beta_{k-1}(s')$

4. **Finally, all the calculated values of $\gamma_k(s',s)$, $\alpha_k(s)$ and $\beta_{k-1}(s')$ are used to calculate the condition LLR function**

$$L(u_k|\tilde{y}) = \ln \frac{\sum_{(s',s) \Rightarrow u_k=+1} \alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}{\sum_{(s',s) \Rightarrow u_k=-1} \alpha_{k-1}(s')\ \gamma_k(s',s)\ \beta_k(s)}$$