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UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

MSc and EEE/ISE PART IV: M.Eng. and ACGI

ADVANCED DATA COMMUNICATIONS

Tuesday, 7 May 10:00 am

There are FOUR questions on this paper.

Answer THREE questions.

Corrected Copy

Time allowed: 3:00 hours

Examiners responsible:

First Marker(s): Gurcan,M.K.

Second Marker(s): Constantinides,A.G.

Special Instructions for Invigilators: None

Information for candidates:

Useful equations

Suppose $g(t)$ and $G(f)$ are Fourier transform pairs such that

$$g(t) \Leftrightarrow G(f)$$

where

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)\,dt \text{ and}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\,df\,.$$

Then the following Fourier transform relationships might be useful

$$g(t) = rect\left(\frac{t}{T}\right) \quad \Leftrightarrow \quad G(f) = T\,\mathrm{sinc}(f\,T)$$

$$g(t) = \frac{1}{2}[\delta(t-t_0)+\delta(t+t_0)] \quad \Leftrightarrow \quad G(f) = \cos(2\pi f t_0)$$

$$g(t) = \sin(2\pi f_c t) \quad \Leftrightarrow \quad G(f) = \frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)].$$

$$g(t) = \delta(t) \quad \Leftrightarrow \quad G(f) = 1$$

$$x(t) = \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1-\frac{4\alpha^2 t^2}{T^2}} \quad \Leftrightarrow \quad X_{RC}(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left(|f|-\frac{1-\alpha}{2T}\right)\right)\right\}, & \frac{1-\alpha}{2T} < |f| \le \frac{1+\alpha}{2T}. \\ 0, & |f| > \frac{1+\alpha}{2T} \end{cases}$$

1. a) A Hadamard matrix is defined as a matrix whose elements are ± 1 and has row vectors which are pairwise orthogonal. In the case when n is a power of 2, an $n \times n$ Hadamard matrix is constructed by means of the recursion

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

i) Let c_i denote the i^{th} row of an $n \times n$ Hadamard matrix as defined above. Show that the waveforms constructed as [3]

$$s_i(t) = \sum_{k=1}^{n} c_{ik}\, p(t - k\,T_c), \qquad i = 1,2.........n$$

are orthogonal, where $p(t)$ is an arbitrary pulse confined to the time interval $0 \le t \le T_c$.

ii) Show that the n matched filters (or cross correlators) for the n waveforms $\{s_i(t)\}$ can be realised by a single filter (or correlator) matched to the pulse $p(t)$ followed by n correlators using the code words $\{c_i\}$. [3]

b) The discrete sequence

$$r_k = \sqrt{\varepsilon_b}\, c_k + n_k, \qquad k = 1,2.........n$$

represents the output sequence of samples from a demodulator, where $c_k = \pm 1$ are elements of one of two possible codewords $c_1 = [1,1,1,...........1]$ and $c_2 = [1,1,1,......,-1,-1-1,-1.....-1]$. The codeword c_2 has w elements which are $+1$ and n-w elements which are -1, where w is some positive integer. The noise sequence $\{n_k\}$ is white Gaussian with variance σ^2. The term ε_b is the bit energy.

i) What is the optimum maximum-likelihood sequence detector for the two possible transmitted signals? [3]

ii) Determine the probability of error as a function of parameter $(\sigma^2, \varepsilon_b, w)$. [2]

iii) What is the value of w that minimises the error probability? [2]

c) Consider the four-phase and eight-phase signal constellations shown in Figure 1.1. Determine the radii r_1 and r_2 of the circles such that the distance between two adjacent points in the two constellations is d. From this result, determine the additional transmitted energy required in the 8-PSK signal to achieve the same error probability as the four-phase signal at high SNR where the probability of error is determined by errors in selecting adjacent points. [4]

Question continued over

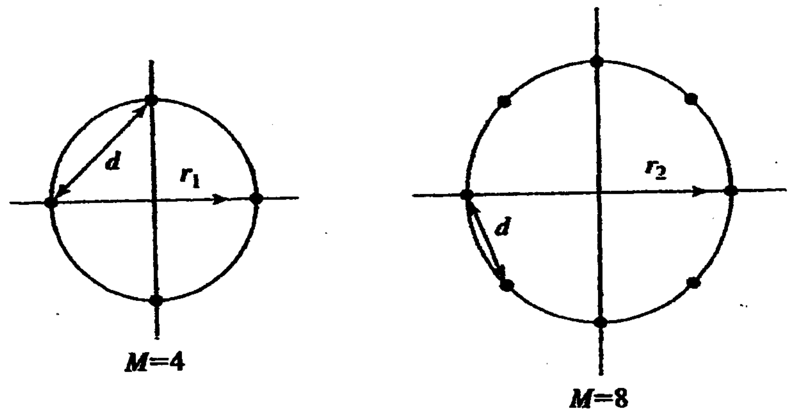


Figure 1.1

d) In a MSK signal, the initial state for the phase is either 0 or π radians. Determine the terminal phase state for the following four input pairs of input data (a) 00 , (b) 01, (c) 10, (d) 11. [3]

2. a) Consider a four phase PSK signal that is represented by the equivalent lowpass signal.

$$v(t) = \sum_n a_n \, g(t - nT)$$

where a_n takes on one of the four possible values $\frac{\pm 1 \pm j}{\sqrt{2}}$ with equal probability.

The sequence of information symbols $\{a_n\}$ is statistically independent.

i) Determine and sketch the power-spectral density of $v(t)$ when [3]

$$g(t) = \begin{cases} A, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}.$$

ii) Repeat part i) when [3]

$$g(t) = \begin{cases} A \sin\frac{\pi t}{2}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}.$$

b) The frequency-response characteristics of a low-pass channel can be approximated by

$$H(f) = \begin{cases} 1 + \alpha \cos 2\pi f t_0, & |\alpha| < 1, \quad |f| \le W \\ 0, & \text{otherwise} \end{cases}$$

where W is the channel bandwidth. An input signal $s(t)$ whose spectrum is band limited to W Hz is passed through the channel.

i) Show that the filtered signal is given by $y(t) = s(t) + \frac{\alpha}{2}\left[s(t - t_0) + s(t + t_0)\right]$. [2]

ii) Suppose the received signal $y(t)$ is passed through a filter matched to $s(t)$. Determine the output of the matched filter at $t = kT$, $k = 0, \pm 1, \pm 2, \ldots\ldots\ldots$ where T is the symbol duration. [2]

iii) What is the ISI pattern resulting from the channel if $t_0 = T$? [2]

Question continued over

c) Show that a pulse having the raised cosine spectrum given by [3]

$$X_{RC}(f)=\begin{cases} T, & 0\le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left\{1+\cos\left(\frac{\pi T}{\alpha}\left(|f|-\frac{1-\alpha}{2T}\right)\right)\right\}, & \frac{1-\alpha}{2T}\le |f| \le \frac{1+\alpha}{2T} \\ 0, & |f| > \frac{1+\alpha}{2T} \end{cases}$$

satisfies the Nyquist criterion given by equation

$$x(nT)=\begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

for any value of the roll-off factor α. Here n is an integer.

d) An M-ary Pulse Amplitude Modulation communication system transmits data at a rate of 4800 symbols/s over a channel with frequency (magnitude) response

$$|C(f)|=\frac{1}{\sqrt{1+\left(\frac{f}{W}\right)^2}} \quad \text{for } |f|\le W$$

where W=4800 Hz. The additive noise is zero-mean, white, Gaussian with power spectral density $N_0/2=10^{-15}$ W/Hz.

i) Suppose that we select the filter at the transmitter to have the frequency response $G_T(f)=\frac{\sqrt{X_{RC}(f)}}{C(f)}\exp(-j2\pi f t_0)$. Here t_0 is a suitable delay to ensure casuality. Determine the magnitude of the transmitting and receiving filter characteristics. [3]

ii) Assume that the transmitting and receiving filters satisfy [2]

$|G_T(f)||G_R(f)|=X_{RC}(f)$.

If the channel, $C(f)$, is equalised by a zero forcing equaliser, determine the value of the noise variance at the sampling instants and the probability of error.

3 a) The convolution encoder shown in Figure 3.1 is used to send the data sequence

1 1 0 1 1

over a binary symmetric channel.

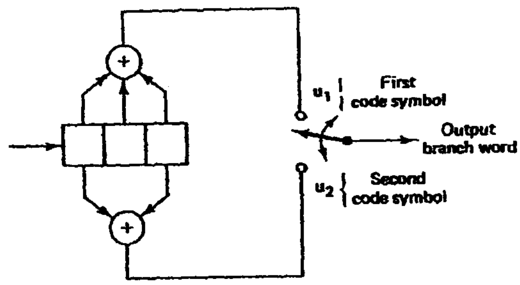


Figure 3.1

Assume that the encoder is initialised with data 0 0 0.

i) Draw the trellis diagram for the encoder. [3]

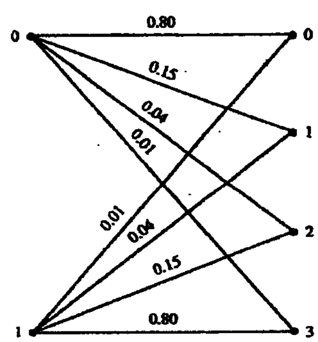
ii) Produce the transmitted codeword. [3]

iii) If the received sequence is [5]

11 01 01 10 01

using the Viterbi algorithm identify the surviving paths over a five symbol period and hence show how the Viterbi algorithm corrects the received sequence.

b) The truncated encoder given in part a is used in a soft decision system. The binary input discrete multi channel for this system is shown in Figure 3.2 together with the channel transition probabilities along with log-likelihood metrics for all the transitions.



$$\ln[\Pr(0|0)] = \ln[\Pr(3|1)] = \ln[0.80] = -0.22$$

$$\ln[\Pr(1|0)] = \ln[\Pr(2|1)] = \ln[0.15] = -1.90$$

$$\ln[\Pr(2|0)] = \ln[\Pr(1|1)] = \ln[0.04] = -3.22$$

$$\ln[\Pr(3|0)] = \ln[\Pr(0|1)] = \ln[0.01] = -4.61$$

Figure 3.2

Use the trellis description of the encoder and the soft decision received sequence

32 21 33 32 02 32 03 to

i) Produce the branch metrices and surviving paths over six symbol periods. [4]

ii) Decode the soft decision received sequence. [5]

4 a) A trellis-coded modulation system uses an 8-ary PAM signal set given by $\{\pm 1, \pm 3, \pm 5, \pm 7\}$ and the 4-state trellis encoder shown in Figure 4.1

i) Using the set partitioning rules, partition the signal set into four subsets. [3]

ii) If the channel is additive white Gaussian noise, and at the output of the matched filter the sequence $(-0.2, 1.1, 6, 4, -3, -4.8, 3.3)$ is observed, what is the most likely transmitted sequence? [5]

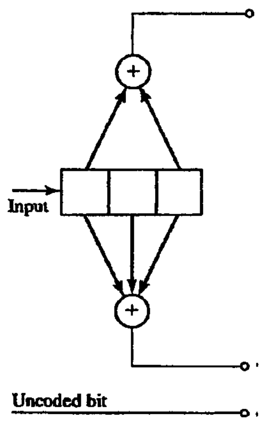


Figure 4.1

b) Design an OFDM system with the following requirements [5]

- Bit rate 30Mbps.
- Tolerable delay spread 400 ns.
- Bandwidth less than 13 MHz.

c) Suppose that a turbo encoder is described by the trellis shown in Figure 4.2 where we assume that the depth of the trellis for the received sequence is N stages and we are considering stage i.

i) Describe how the BJCR algorithm is used recursively to calculate the log-likelihood ratios. [5]

ii) Describe how the decoder bit error rate performance improves as the number of iterations increases. [2]

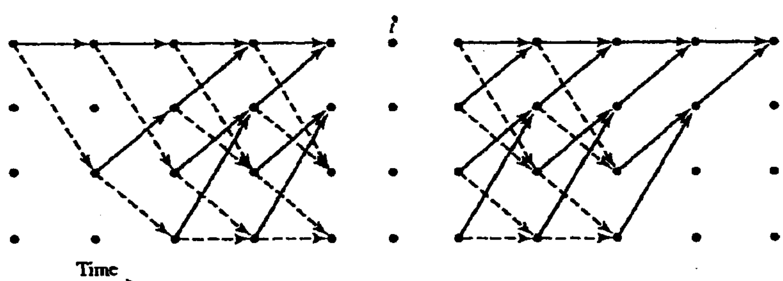


Figure 4.2

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i) The inner product of $s_i(t)$ and $s_j(t)$ is

$$\int_{-\infty}^{\infty} s_i(t)\, s_j(t)\,dt = \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik} p(t-kT_c) \sum_{l=1}^{n} c_{jl} p(t-lT_c)\,dt$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik} c_{jl} \int_{-\infty}^{\infty} p(t-kT_c) p(t-lT_c)\,dt$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} c_{ik} c_{jl} \mathcal{E}_p \delta_{kl} = \mathcal{E}_p \sum_{k=1}^{n} c_{ik} c_{jk}$$

The quantity $\sum_{k=1}^{n} c_{ik} c_{jk}$ is the inner product of the row vectors $\underline{c}_i$ and $\underline{c}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)\, s_j(t)\,dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2 \delta_{ik} = n\, \mathcal{E}_p\, \delta_{ij}$$

Thus the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

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ii) First we consider the signal

$$y(t) = \sum_{k=1}^{n} c_k\, \delta(t-kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is

$$g(t) = y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_n\, \delta(nT_c - kT_c - t)$$

$$= \sum_{i=1}^{n} c_{n-i+1}\, \delta((i-1)T - t) = \sum_{i=1}^{n} c_{n-i+1}\, \delta(t-(i-1)T)$$

that is a sequence of impulses starting at $t=0$ and weighted by the mirror image sequence of $\{c_i\}$. Since

$$s(t) = \sum_{k=1}^{n} c_k\, p(t-kT_c) = p(t) * \sum_{k=1}^{n} c_k\, \delta(t-kT_c)$$

the Fourier transform of the signal $s(t)$ is

$$S(f) = P(f) \sum_{k=1}^{n} c_k \exp(-j 2\pi f k T_c)$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$H(f) = S^*(f) \exp(-j2\pi f T) = S^*(f) \exp(-j2\pi f nT_c)$$

$$= P^*(f) \sum_{k=1}^{n} c_k \exp(j 2\pi f k T_c) \exp(-j2\pi f n T_c)$$

$$= P^*(f) \sum_{i=1}^{n} c_{n-i+1} \exp(-j2\pi f (i-1)T_c)$$

$$= P^*(f)\, \mathcal{F}[g(t)]$$

Thus the matched filter $H(f)$ can be considered as the cascade of a filter, with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal

$$y(t) = \sum_{k=1}^{n} c_k\, \delta(t-kT_c).$$

The output of the matched filter at $t = nT_c$ is

$$\int_{-\infty}^{\infty} |s(t)|^2 = \sum_{k=1}^{n} c_k \int_{-\infty}^{\infty} p^2(t - kT_c)\,dt$$

$$= T_c \sum_{k=1}^{n} c_k^2$$

where we have used the fact $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Using the above result we obtain the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik}\, p(t - kT_c)$$

can be realised as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{c}_i = [c_{i,1}, \cdots c_{i,n}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$ we conclude that the n matched filters can be realised by a filter matched to $p(t)$ followed by n discrete-time filters matched to vectors $\underline{c}_i$, $i = 1 \cdots n$.

i) The optimal maximum likelihood selects the sequence $\underline{c}_i$ that minimizes the quantity

$$\lambda_i = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b}\, c_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$\lambda_1 = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

$$\lambda_2 = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first terms of the right side of the two equations are common, we conclude that the optimal maximum likelihood detector can base its decisions only on the last $n-w$ received elements of r. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \underset{c_1}{\overset{c_2}{\gtrless}} 0$$

Or equivalently

$$\sum_{k=w+1}^{n} r_k \underset{c_2}{\overset{c_1}{\gtrless}} 0$$

ii) Since $r_k = \sqrt{\mathcal{E}_b}\, c_{ik} + n_k$, the probability of error $P(e|\underline{c}_1)$ is

$$P(e|\underline{c}_1) = P\left(\sqrt{\mathcal{E}_b}\,(n-w) + \sum_{k=w+1}^{n} n_k < 0\right)$$

$$= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b}\right)$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$

Hence

$$P(e|\underline{c}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\mathcal{E}_b(n-w)} \exp\left(-\frac{x^2}{2\pi(n-w)\sigma^2}\right) dx$$

$$= Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

Similarly we find

$P(e|\underline{c}_2) = P(e|\underline{c}_1)$ and since two sequences are equiprobable

$$P(e) = \frac{1}{2}P(e|\underline{c}_2) + \frac{1}{2}P(e|\underline{c}_1) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

iii) The probability of error $P(e)$ is minimised when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximised, that is for $w=0$. This implies that $\underline{c}_1 = -\underline{c}_2$ and thus the distance between the sequences is the maximum possible.

Using the pythagorean theorem for the four-phase constellation, we find

$$r_1^2 + r_1^2 = d^2 \Rightarrow r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus

$$d^2 = r_2^2 + r_2^2 - 2r_2^2\cos(45°) = \quad r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}$$

Thus the additional transmitted power needed by the 8-PSK signal is

$$P = 10\log_{10}\frac{2d^2}{(2-\sqrt{2})d^2} = 5.3329\,\text{dB}$$

We obtain the same results if we use the probability of error given by

$$P_M = 2Q\left[\sqrt{2\rho_2}\,\sin\frac{\pi}{M}\right]$$

where ρ_s is the SNR per symbol. In this case, equal error probability for the two signalling schemes implies that

$$\rho_{4,s}\sin^2\frac{\pi}{4} = \rho_{8,s}\sin^2\frac{\pi}{8} \Rightarrow 10\log_{10}\frac{\rho_{8,s}}{\rho_{4,s}}$$

$$= 20\log_{10}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}} = 5.3329\,\text{dB}.$$

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant n is given by

$$\theta(n;a) = \frac{\pi}{2}\sum_{k=0}^{k} a_k + \theta_0$$

Where θ_0 is the initial and a_k is ± 1 depending on the input bit at the time instant k. The following table shows $\theta(n;a)$ for two different values of θ_0 (0, π), and the four input pairs of data {00, 01, 10, 11}

θ_0	b_0	b_1	a_0	a_1	$\theta(n;a)$
0	0	0	-1	-1	-π
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

i) The autocorrelation function of the information symbols $\{a_n\}$ is

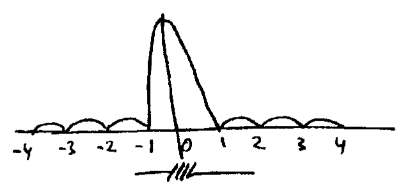
$$R_a(k) = E\left(a_n^* a_{n+k}\right) = \frac{1}{4} \times |a_n|^2 \delta(k) = \delta(k)$$

Thus, the power spectral density of v(t) is

$$S_v(f) = \frac{1}{T} S_a(f) \quad |G(f)|^2 = \frac{1}{T}|G(f)|^2$$

Where $G(f) = \mathfrak{F}(g(t))$. If $g(t) = A\cdot \mathrm{rect}\left(\frac{t - T/2}{T}\right)$, we obtain $|G(f)|^2 = A^2T^2 \mathrm{sinc}^2(fT)$ and therefore,

$$S_v(f) = A^2 T\,\mathrm{sinc}^2(fT)$$



ii) If $g(t) = A\cdot \sin\left(\frac{\pi t}{2}\right)\mathrm{rect}\left(\frac{t - T/2}{T}\right)$ then.

$$G(f) = A\left[\frac{1}{2j}\delta\left(f-\frac{1}{4}\right) - \frac{1}{2j}\delta\left(f+\frac{1}{4}\right)\right] * T\,\mathrm{sinc}(fT)\exp\left(-j2\pi f\frac{T}{2}\right)$$

$$= \frac{AT}{2}\left[\delta\left(f-\frac{1}{4}\right) - \delta\left(f+\frac{1}{4}\right)\right] * \mathrm{sinc}(fT)\exp\left(-j\left(2\pi f\frac{T}{2} + \frac{\pi}{2}\right)\right)$$

$$= \frac{AT}{2}\exp\left(-j\pi\left(f-\frac{1}{4}\right)T + \frac{1}{2}\right)\left[\mathrm{sinc}\left(\left(f-\frac{1}{4}\right)T\right) - \mathrm{sinc}\left(\left(f-\frac{1}{4}\right)T\right)\exp\left(-j\frac{\pi}{2}\right)\right]$$

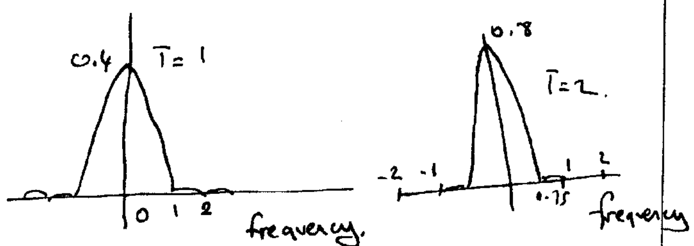
Thus

$$|G(f)|^2 = \frac{A^2T^2}{4}\left[\mathrm{sinc}^2\left(f+\frac{1}{4}\right)T + \mathrm{sinc}^2\left(f-\frac{1}{4}\right)T - 2\,\mathrm{sinc}\left(\left(f+\frac{1}{4}\right)T\right)\mathrm{sinc}\left(f-\frac{1}{4}\right)T\cos\frac{\pi T}{2}\right]$$

and the power spectral density of the transmitted signal is

$$S_v(f) = \frac{A^2T}{2}\Big[\mathrm{sinc}^2\big((f+\tfrac{1}{4})T\big) + \mathrm{sinc}^2\big((f-\tfrac{1}{4})T\big) - 2\,\mathrm{sinc}\big((f+\tfrac{1}{4})T\big)\,\mathrm{sinc}\big((f-\tfrac{1}{4})T\big)\cos\frac{\pi T}{2}\Big]$$

The plot of $S_v(f)$ for two special values of the time interval T. The amplitude of the signal A was set to 1 for both cases.



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i) Taking the inverse Fourier transform of $H(f)$, we obtain

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \frac{\alpha}{2}\delta(t-t_0) + \frac{\alpha}{2}\delta(t+t_0)$$

Hence

$$y(t) = s(t) * h(t) = s(t) + \frac{\alpha}{2}s(t-t_0) + \frac{\alpha}{2}s(t+t_0)$$

—///—

ii) If the signal $s(t)$ is used to modulate the sequence $\{a_n\}$, then the transmitted signal

$$u(t) = \sum_{n=-\infty}^{\infty} a_n\, s(t-nT)$$

The received signal is the convolution of $u(t)$ with $h(t)$. Hence

$$y(t) = u(t) * h(t) = \Big(\sum_{n=-\infty}^{\infty} a_n\, s(t-nT)\Big) * \Big(\delta(t) + \frac{\alpha}{2}\delta(t-t_0) + \frac{\alpha}{2}\delta(t+t_0)\Big)$$

$$= \sum_{n=-\infty}^{\infty} a_n\, s(t-nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n\, s(t-t_0-nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n\, s(t+t_0-nT)$$

Thus, the output of the matched filter $s(t)$ at the time instant t_1 is

$$w(t_1) = \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau-nT)\, s(\tau-t_1)\, d\tau + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau-t_0-nT)\, s(\tau-t_1)\, d\tau + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau+t_0-nT)\, s(\tau-t_1)\, d\tau.$$

If we denote the signal $s(t) * s(t)$ by $x(t)$, then the output of the matched filter at $t_1 = kT$ is

$$w(kT) = \sum_{n=-\infty}^{\infty} a_n x(kT - nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n x(kT - t_0 - nT) + \frac{\alpha}{2}\sum_{n=-\infty}^{\infty} a_n x(kT + t_0 - nT)$$

iii) With $t_0 = T$ and $k = n$ in the previous equation, we obtain

$$w_k = a_k x_0 + \sum_{n \neq k} a_n x_{k-n} + \frac{\alpha}{2} a_k x_{k-1} + \frac{\alpha}{2}\sum_{n\neq k} a_n x_{k-n-1} + \frac{\alpha}{2} a_k x_1 + \frac{\alpha}{2}\sum_{n \neq k} a_n x_{k-n+1}$$

$$= a_k\left(x_0 + \frac{\alpha}{2}x_{-1} + \frac{\alpha}{2}x_1\right) + \sum_{n\neq k} a_n\left[x_{k-n} + \frac{\alpha}{2}x_{k-n-1} + \frac{\alpha}{2}x_{k-n+1}\right]$$

The term under the summation is the ISI introduced by the channel.

The pulse $x(t)$ having the raised cosine spectrum is

$$x(t) = \operatorname{sinc}(t/T)\,\frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1 - 4\alpha^2 t^2/T^2}$$

The function $\operatorname{sinc}\left(\frac{t}{T}\right)$ is 1 when $t=0$ and 0 when $t = nT$. On the other hand

$$g(t) = \frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1 - \frac{4\alpha^2 t^2}{T^2}} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $\frac{4\alpha^2 t^2}{T^2} = 1$ or $\alpha t = \frac{T}{2}$.

However

$$\lim_{\alpha t \to \frac{T}{2}} \frac{\cos\left(\frac{\pi\alpha t}{T}\right)}{1 - \frac{4\alpha^2 t^2}{T^2}} = \lim_{x\to 1}\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x}$$

and by using L'Hospital rule

$$\lim_{x\to 1}\frac{\cos\left(\frac{\pi}{2}x\right)}{1-x} = \lim_{x\to 1}\frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = \frac{\pi}{2} < \infty$$

Hence

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

i) Since $W = \frac{1}{T} = 4800$, we use a signal pulse with a raised cosine spectrum and $\alpha = 1$. Thus

$$X_{RC}(f) = \frac{T}{2}\left[1 + \cos(\pi T|f|)\right] = T\cos^2\left(\frac{\pi|f|}{9600}\right)$$

Then

$$|G_T(f)| = \sqrt{T\left[1+\left(\frac{f}{W}\right)^2\right]}\cos\frac{\pi|f|}{9600}, \quad |f| \leq 4800\,\text{Hz}$$

$$|G_R(f)| = T.\cos\frac{\pi|f|}{9600}, \quad |f| \leq 4800\,\text{Hz}.$$

and $|G_T(f)| = |G_R(f)| = 0$ for $|f| > 4800\,\text{Hz}$.

ii) when the noise is white, the variance of the noise at the output of zero forcing equaliser is given by

$$\sigma_v^2 = \frac{N_0}{V}\int_{-W}^{W}\frac{X_{RC}(f)}{|C(f)|^2}\,df = \frac{TN_0}{2}\int_{-W}^{W}\left[1+\left(\frac{f}{W}\right)^2\right]\cos^2\frac{\pi|f|}{2W}\,df$$

$$= N_0\int_0^1 (1+x^2)\cos^2\frac{\pi x}{2}\,dx$$

$$= \left(\frac{2}{3} - \frac{1}{\pi^2}\right)N_0$$

The average transmitted power is

$$P_{av} = \frac{(M^2-1)d^2}{3T}\int_{-W}^{W}|G_T(f)|^2\,df.$$

$$= \frac{(M^2-1)d}{3T}\int_{-\infty}^{\infty}|X_{RC}(f)|\,df.$$

$$= \frac{(M^2-1)d^2}{3T}$$

The general expression for the probability of error is given as

$$P_M = \frac{2(M-1)}{M}\,Q\left(\sqrt{\frac{3P_{av}T}{(M^2-1)\left(\frac{2}{3}-1/\pi^2\right)N_0}}\right)$$

If the channel is ideal, the argument of the Q function would be $6P_{av}T/(M^2-1)N_0$

Hence the loss in performance due to the non-ideal channel is given by the factor $2\left(\frac{2}{3} - \frac{1}{\pi^2}\right) = 1.113$ or $0.54\,\text{dB}$.

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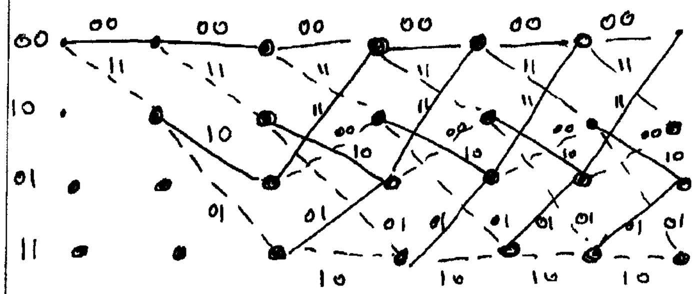
MODEL ANSWER and MARKING SCHEME

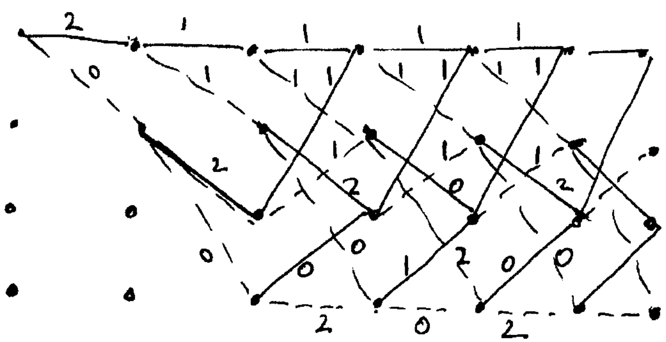
First Examiner	Dr. M.K. Gurcan	**Paper Code**	
Second Examiner	Prof. A.G. Constatinides	**Question** 3a	**Page** 15 *out of*
Question labels in left margin		**Marks allocations in right margin**	

i)



ii)

Input	1	1	0	1	1
Output	11	01	01	00	01
Received	11	01	01	10	01



Difference trellis.

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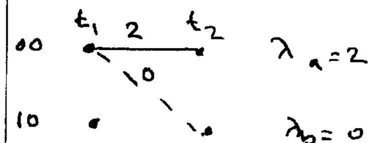
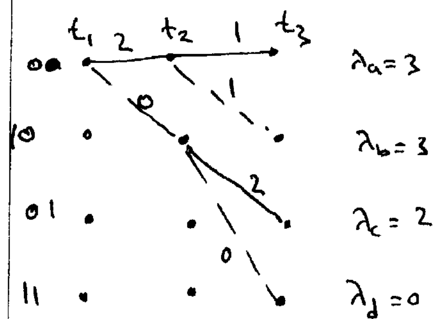
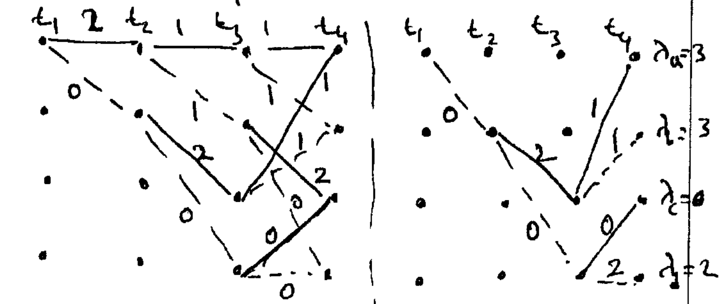
Path metric at the end of first time slot



At the end of second time slot



At the end of third time slot



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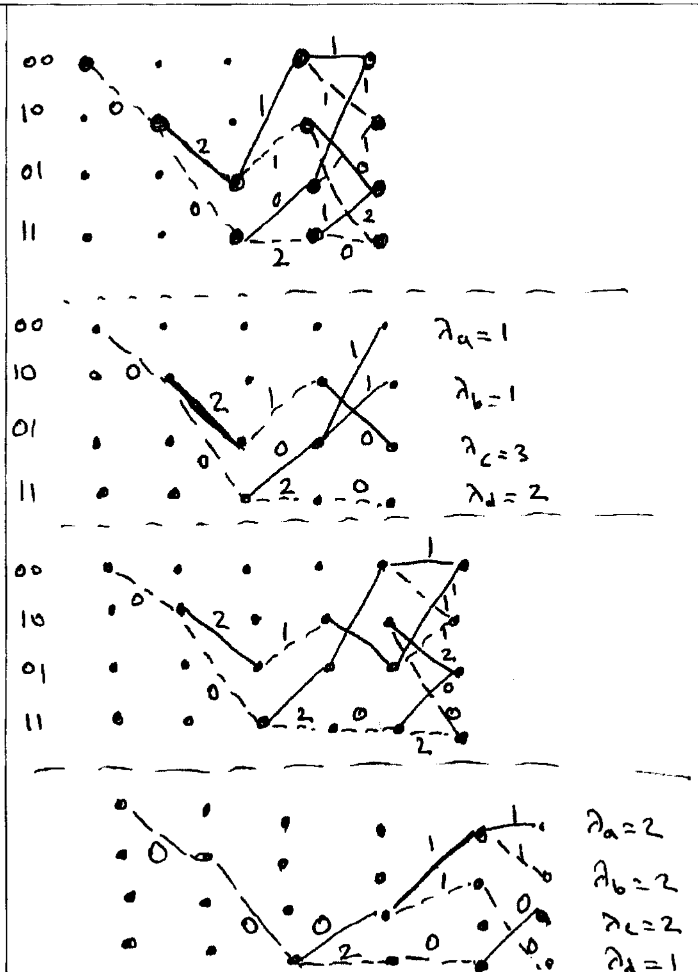

Received Sequence. 1 1 0 1 1

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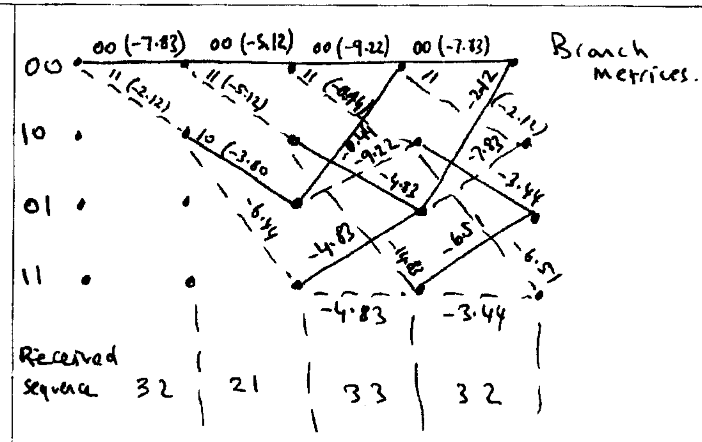
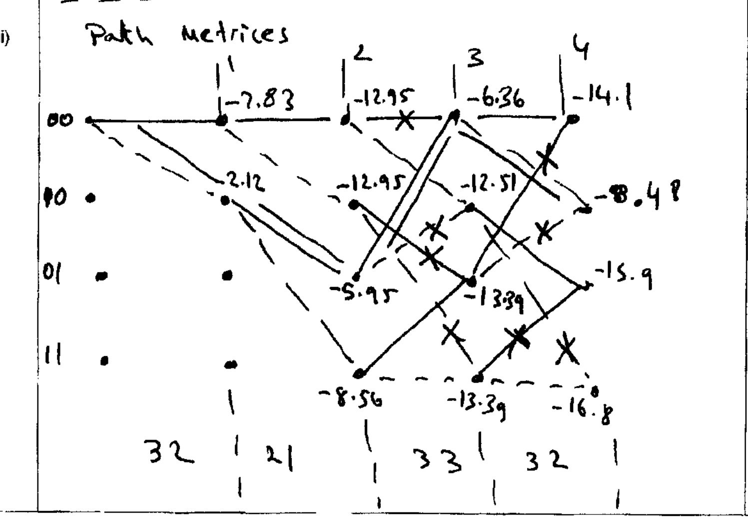
Question labels in left margin | **Marks allocations in right margin**

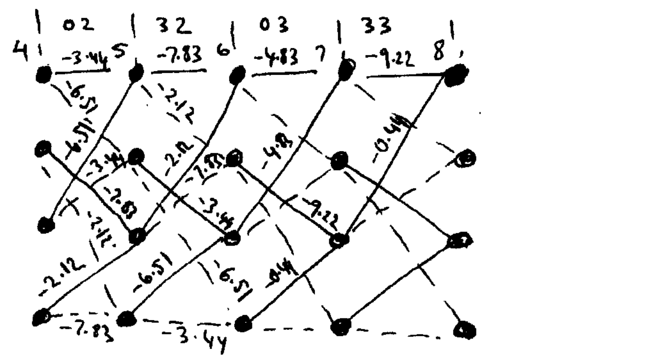
i)

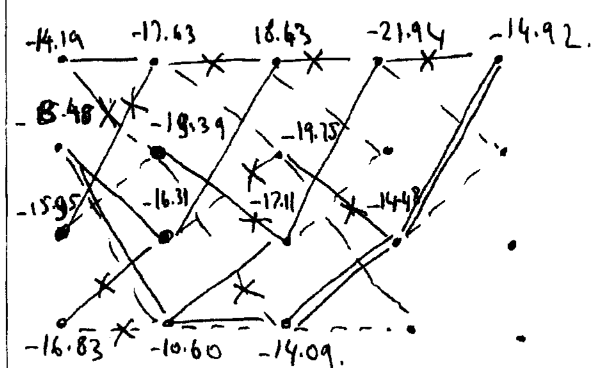


ii)



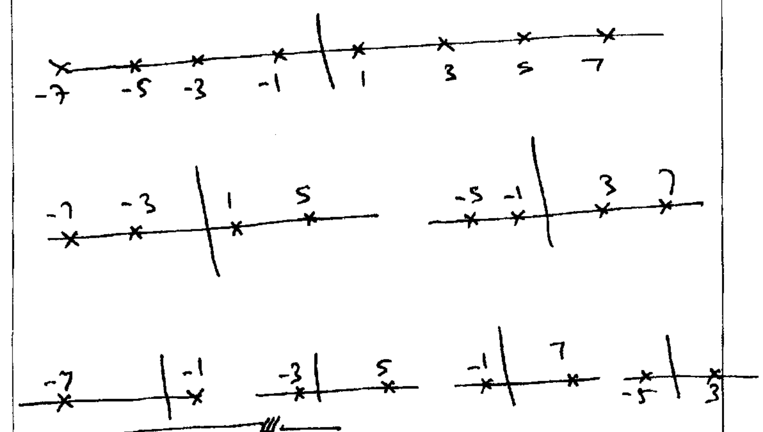
Received data 1 0 0 1





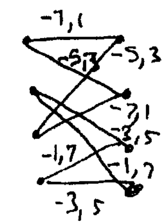
1 1 0 0

The partion of the 8 PAM constellation in four subsets is undertaken as follows

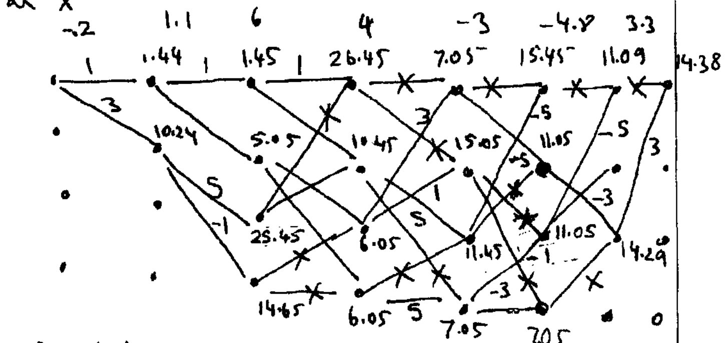


ii

The next figure shows one frame of the trellis used to decode the received sequence. Each branch consists of two transitions which correspond to elements in the same coset in the final partion level.



The next figure gives the operation of the Viterbi algorithm for the decoding of the sequence (−0.2 , 1.1 , 6, 4 , −3 , −4.8, 3.3)
We assume that we start at the all zero state and that a sequence of zeros terminates the input bit stream in order to clear the encoder, The numbers at the nodes indicate the minimum Euclidean distance, and the branches have been marked with the transmitted symbol. The paths that have been purged are marked with an X



Decoded sequence

1 3 5 3 -5 -3 3

Delay spread = 400 ns.

Guard time = 4x delay spread = 1.6 μs.

Symbol duration = $6 \times 1.6 = 9.6\ \mu s$.

Sub carrier spacing = $\dfrac{1}{(9.6-1.6)10^{-6}} = 125\ \text{kHz}$

Number of bits per symbol

$$x \cdot \frac{1}{9.6 \cdot 10^{-6}} = 30 \cdot 10^{6}$$

$x = 288$ bits.

Use 96 carriers and rate 3/4 to get the required number of bits per symbol

$$\frac{3}{4}\, 96 \cdot n = 288$$

$n = 4$ bits per carrier

$2^4 = 16$ QAM is required.

Required transmission bandwidth

$96 \times 125 = 12$ MHz < 13 MHz max bandwidth.

i) Turbo algorithm.

1.) initialize the algorithm with boundary conditions
$\alpha(0) = B(N) = 0$ for all states. except start and finish states.

2.) given prior probabilities of $u(i)$, the channel characteristics and the received sequence calculate

$$\gamma_k(s',s) = f(\bar{y}_k \mid \bar{u}_k)\, P(u_{k,i})$$

if LLR (u_k) is given

$$P(u_k = \pm 1) = \frac{\exp(-L(u_k)/2)}{1+\exp(-L(u_k))} \exp\left(u_k \frac{L(u_k)}{2}\right)$$

also we have

$$f(\bar{y}_k \mid \bar{u}_k) = \prod_{l=1}^{m} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0}(y_{kl} - u_{kl})^2\right)$$

3.) Calculate $\alpha_k(s')$ and $\beta_{k-1}(s)$ using

$$\alpha_k(s) = \sum_{\text{all } s'} \gamma(s',s)\, \alpha_{k-1}(s)$$

$$\beta_{k-1} = \sum_{\text{all } s'} \gamma_k(s',s)\, \beta_k(s)$$

4) calculate the LLR for u_k.

$$L(u_k \mid \bar{y}) = \ln \frac{\sum_{(s',s) \Rightarrow u_k = +1} \alpha_{k-1}(s')\, \gamma_k(s',s)\, \beta_k(s)}{\sum_{(s',s) \Rightarrow u_k = -1} \alpha_{k-1}(s')\, \gamma_k(s',s)\, \beta_k(s)}$$

5) From 4) calculate extrinsic LLR

$$LLR = L_e(u_k) = L(u_k \mid \bar{y}) - L(u_k) - L_c y_{ks}$$

$L(u_k)$ = a-priori information about u_k.

6) $L_e(u_k)$ becomes a-priori LLR and the sequence is repeated

ii)