UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE TECHNOLOGY AND MEDICINE [MSc-C2, E401, I411]

DEPARTMENT of ELECTRICAL and ELECTRONIC ENGINEERING
EXAMINATIONS 2001
M.Sc in Communications and Signal Processing
M.Eng. Part IV

Solutions 2001
ADVANCED COMMUNICATION THEORY

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ANSWER to Q1

1)	A	B	C	D	E
2)	A	B	C	D	E
3)	A	B	C	D	E
4)	A	B	C	D	E
5)	A	B	C	D	E
6)	A	B	C	D	E
7)	A	B	C	D	E
8)	A	B	C	D	E
9)	A	B	C	D	E
10)	A	B	C	D	E
11)	A	B	C	D	E
12)	A	B	C	D	E
13)	A	B	C	D	E
14)	A	B	C	D	E
15)	A	B	C	D	E
16)	A	B	C	D	E
17)	A	B	C	D	E
18)	A	B	C	D	E
19)	A	B	C	D	E
20)	A	B	C	D	E

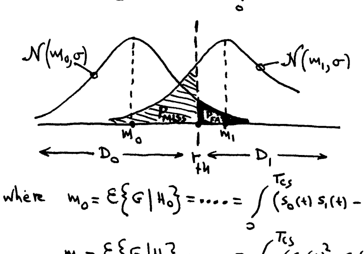
ANSWER to Q2

The correlation receiver implements the following optimum decision rule:

$$\boxed{\text{choose } H_1 \ \underline{\underline{\text{iff}}} \ G \geq r_{th}}$$

where $G \triangleq \int_0^{T_{cs}} r(t).s_1(t)\,dt - \int_0^{T_{cs}} r(t).s_0(t)\,dt$

$$r_{th} \triangleq \frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\int_0^{T_{cs}} (s_1(t)^2 - s_0(t)^2)\,dt$$



where $m_0 = \mathcal{E}\{G|H_0\} = \cdots = \int_0^{T_{cs}} (s_0(t)\,s_1(t) - s_0(t)^2)\,dt$

$$m_1 = \mathcal{E}\{G|H_1\} = \cdots = \int_0^{T_{cs}} (s_1(t)^2 - s_0(t)\,s_1(t))\,dt$$

$\sigma^2 = \mathrm{Var}\{G\}$ ← for any Hypothesis

$$= \cdots = \frac{N_0}{2}\int_0^{T_{cs}} (s_1(t)^2 + s_0(t)^2 - 2s_1(t)s_0(t))\,dt$$

$$= N_0 E(1-\rho)$$

$$\therefore\ P_e = Pr(H_0).\underbrace{Pr(D_1|H_0)}_{=P_{FA} = T\left\{\frac{r_{th}-m_0}{\sigma}\right\}} + Pr(H_1).\underbrace{Pr(D_0|H_1)}_{=P_{MISS} = T\left\{\frac{r_{th}-m_1}{\sigma}\right\}}$$

However $\frac{r_{th}-m_0}{\sigma}$ can be express as a function of λ_0, EUE, ρ as follows:

$$\frac{r_{th}-m_0}{\sigma} = \frac{\overbrace{\frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\int_0^{T_{cs}}(s_1(t)^2 - s_0(t)^2)\,dt}^{=r_{th}} - \overbrace{\int_0^{T_{cs}}(s_0(t)s_1(t) - s_0(t)^2)\,dt}^{=m_0}}{\underbrace{\sqrt{N_0E(1-\rho)}}_{=\sigma}}$$

$$= \frac{\frac{N_0}{2}\ln(\lambda_0) + \frac{1}{2}\overbrace{\int_0^{T_{cs}}(s_1(t)^2 + s_0^2(t))\,dt}^{=E} - \overbrace{\int_0^{T_{cs}} s_0(t).s_1(t)\,dt}^{=\rho E}}{\sqrt{N_0E(1-\rho)}}$$

$$= \frac{\frac{N_0}{2}\ln(\lambda_0) + E(1-\rho)}{\sqrt{N_0E(1-\rho)}} = \frac{\frac{1}{2}\ln(\lambda_0)}{\sqrt{(1-\rho)EUE}} + \sqrt{(1-\rho)EUE}$$

similarly, it can be found that

$$\frac{r_{th}-m_1}{\sigma} = \frac{\frac{1}{2}\ln(\lambda_0)}{\sqrt{(1-\rho)EUE}} - \sqrt{(1-\rho)EUE}$$

$\therefore$ Equ-1 (above) becomes

$$\boxed{P_e = Pr(H_0).\,T\left(\frac{\frac{1}{2}\ln(\lambda_0)}{A} + A\right) + Pr(H_1)\,T\left(\frac{\frac{1}{2}\ln(\lambda_0)}{A} - A\right) \quad \text{Where } A = \sqrt{(1-\rho)EUE}}$$

* Bayes' decision criterion: if $\lambda(\underline{r}) \underset{=}{>} \lambda_0$ $\begin{cases}\text{then choose } H_1\\ \text{otherwise } H_0\end{cases}$

where $\lambda_0 = \frac{\Pr(H_0)}{\Pr(H_1)} \cdot \frac{C_{10}-C_{00}}{C_{01}-C_{11}}$

$$\Rightarrow \lambda_0 = \frac{1/3}{2/3} \; \frac{1.858}{0.5} \Rightarrow \boxed{\lambda_0 = 1.858}$$

* $P_{MISS} = \Pr(D_0|H_1) = \mathcal{T}\left\{\frac{\frac{1}{2}\ln\lambda_0}{A} - A\right\} = 0.04$

$\Rightarrow$ (from Tail function graph) $\Rightarrow \frac{\frac{1}{2}\ln\lambda_0}{A} - A = 1.75$

$\Rightarrow A^2 + 1.75A - \frac{1}{2}\ln\lambda_0 = 0$ with $A>0$

$\Rightarrow A = \frac{-1.75 \pm \sqrt{1.75^2 + 4\frac{1}{2}\ln\lambda_0}}{2} = 0.162$

$\Rightarrow \sqrt{(1-\rho)\,EUE} = 0.162 \Rightarrow (1-\rho)\underbrace{EUE}_{5.25\times10^{-2}} = 0.162^2$

$\Rightarrow \boxed{\rho = \frac{1}{2}}$

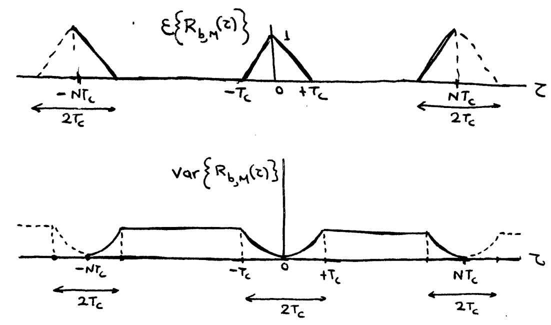
* $P_{FA} = \Pr(D_1|H_0) = \mathcal{T}\left(\frac{\frac{1}{2}\ln\lambda_0}{A} + A\right) = \mathcal{T}(2.0740) \cong 1.7\times10^{-2}$.

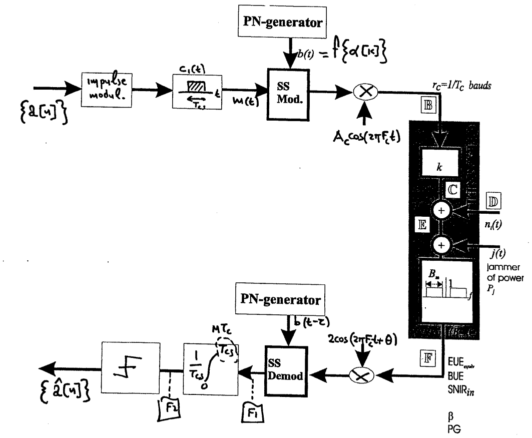
1.858

0.162

$$P_e = \underbrace{\Pr(H_0)}_{1/3} \cdot \underbrace{P_{FA}}_{1.7\times10^{-2}} + \underbrace{\Pr(H_1)}_{\frac{2}{3}} \cdot \underbrace{P_{MISS}}_{4\times10^{-2}} = 3.23\times10^{-2}$$

ANSWER to Q3





$\boxed{B}$: $s(t) = A_c m(t) \cdot b(t) \cos(2\pi F_c t)$

$\boxed{F}$: desired signal term $= ks(t) = \underbrace{kA_c}_{\sqrt{2P_s}} m(t)\, b(t) \cos(2\pi F_c t)$

$\boxed{F1}$: desired signal term $= ks(t) \cdot b(t-\tau)\; 2\cos(2\pi F_c t + \theta)$

$$= \sqrt{2P_s}\, m(t)\, b(t) \cdot b(t-\tau)\; 2\cos(2\pi F_c t)\cos(2\pi F_c t + \theta)$$

$\boxed{F2}$: desired signal term $= m_o(t) =$

$$= \frac{\sqrt{2P_s}}{T_{cs}} \int_0^{T_{cs}} \underbrace{m(t)}_{\pm1} \cdot b(t) \cdot b(t-\tau) \cos\theta \cdot dt$$

$$= \pm \frac{\sqrt{2P_s}}{T_{cs}} \cos\theta \int_0^{MT_c} b(t)\, b(t-\tau)\, dt$$

$$= \pm\sqrt{2P_s} \cos\theta\; R_{b,M}(\tau)$$

Power of $m_o(t) = \mathcal{E}\{m_o^2(t)\} = 2P_s\cos^2\theta \cdot \mathcal{E}\{R_{b,M}^2(\tau)\}$

$$= 2P_s\cos^2\theta \left(\mathrm{var}\{R_{b,M}(\tau)\} + \mathcal{E}^2\{R_{b,M}(\tau)\}\right)$$

$$= \underbrace{2P_s\cos^2\theta\, \mathrm{var}\{R_{b,M}(\tau)\}}_{\text{code noise power}} + \underbrace{2P_s\cos^2\theta\, \mathcal{E}^2\{R_{b,M}(\tau)\}}_{\text{desired term}}$$

1. If $0 \le \tau \le T_c$ then power of code noise $= 2P_s\cos^2\theta\, \mathrm{var}\{R_{b,M}(\tau)\}$

$$= 2P_s\cos^2\theta \cdot \frac{1}{M}\left(\frac{\tau}{T_c}\right)^2$$

$$= 2P_s\cos^2\theta\, \frac{1}{M}\, \frac{\tau^2}{T_c^2} \qquad \left(\text{note}: M = \frac{T_{cs}}{T_c}\right)$$

$$= 2P_s\cos^2\theta\, \frac{1}{T_{cs}T_c} \cdot \tau^2 \qquad \boxed{1}$$

2. If $\tau > T_c$ then power of code noise $= 2P_s\cos^2\theta\, \mathrm{var}\{R_{b,M}(\tau)\}$

$$= 2P_s\cos^2\theta\, \frac{1}{M}$$

$$= 2P_s\cos^2\theta\, \frac{T_c}{T_{cs}} \qquad \boxed{2}$$

$r_c = 10\text{M}\,\frac{\text{chips}}{\text{sec}} \Rightarrow T_c = 10^{-7}$

$r_b = 1000\,\frac{\text{bits}}{\text{sec}} \Rightarrow T_{cs} = 10^{-3}$

$\frac{N_0}{2} = 0.5\times10^{-8} \Rightarrow N_0 = 10^{-8}$

$EUE = 100 \Rightarrow \frac{E_b}{N_0} = 10^2 \Rightarrow \frac{P_s T_{cs}}{N_0} = 10^2 \Rightarrow P_s = 10^2\, \frac{N_0}{T_{cs}} \Rightarrow \boxed{P_s = 10^{-3}}$

10^{-8} 10^{-3}

$P_{\substack{\text{code noise}\\ \tau>T_c}} = 1.5\times10^{-7} \overset{②}{\Longrightarrow} 1.5\times10^{-7} = 2P_s\cos^2\theta\, \frac{T_c}{T_{cs}}$

$$\Rightarrow \cos^2\theta = \frac{1.5\times10^{-7} \cdot T_{cs}}{2P_s\, T_c} \Rightarrow$$

10^{-3} 10^{-3} 10^{-7}

$$\Rightarrow \cos^2\theta = \frac{1.5}{2} = 0.75 \Rightarrow \cos\theta = 0.866$$

$$\Rightarrow \boxed{\theta = 30^\circ}$$

$P_{\substack{\text{code noise}\\ \tau}} = 3.75\times10^{-8} \overset{①}{\Longrightarrow} 3.75\times10^{-8} = 2P_s\, \cos^2\theta\, \frac{1}{T_{cs}\,T_c}\, \tau^2$

10^{-3} 0.75 10^{-3} 10^{-7}

$$\Rightarrow \tau^2 = \frac{3.75\times10^{-15}}{2\times0.75} \Rightarrow \boxed{\tau = 0.5\times10^{-7}} \text{ ie } \boxed{\tau = 0.5\,T_c}$$

ANSWER to Q4

$N_o = 2\times 10^{-6}$ $\quad A_1 = -3\text{mV}$

$T_{cs} = 12$ $\quad A_2 = -1\text{mV}$

$D = 1$ $\quad A_3 = 1\text{mV}$

$M = 4$ $\quad A_4 = 3\text{mV}$

$$PSD_s(f) = \frac{1}{T_{cs}}\,\mathcal{E}\left\{\left|FT\left(A_L\cdot\Lambda\left(\frac{t}{T_{cs}/2}\right)\right)\right|^2\right\}$$

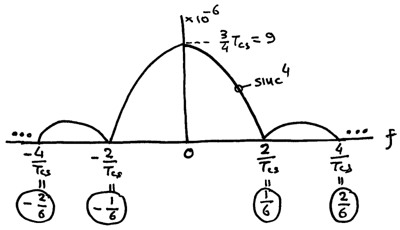
$$= \frac{1}{T_{cs}}\,\mathcal{E}\left\{\left|A_L\frac{T_{cs}}{2}\,\text{sinc}^2\left(f\frac{T_{cs}}{2}\right)\right|^2\right\}$$

$$= \frac{1}{T_{cs}}\,\mathcal{E}\left\{\underbrace{A_L^2}_{\text{random variable}}\frac{T_{cs}^2}{4}\,\text{sinc}^4\left(f\frac{T_{cs}}{2}\right)\right\}$$

$$= \frac{1}{T_{cs}}\cdot\frac{T_{cs}^2}{4}\cdot\text{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\cdot\mathcal{E}\{A_L^2\}$$

$$= \frac{T_{cs}}{4}\,\text{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\left[(-3)^2\frac{1}{8}+(-1)^2\frac{3}{8}+(1)^2\frac{3}{8}+3^2\frac{1}{8}\right]\times10^{-6}$$

$$= \frac{3}{4}\underbrace{T_{cs}}_{12}\,\text{sinc}^4\left(f\cdot\frac{T_{cs}}{2}\right)\times10^{-6} = 9\,\text{sinc}^4(f\,6)\times10^{-6}$$



$$E_L = \int_{-T_{cs}/2}^{T_{cs}/2} A_L^2\,\Lambda^2\left(\frac{t}{T_{cs}/2}\right)dt$$

$$= 2\int_{-T_{cs}/2}^{0} A_L^2\left(\frac{t+T_{cs}/2}{T_{cs}/2}\right)^2 dt$$

$$= 2A_L^2\int_{-T_{cs}/2}^{0}\frac{t^2+T_{cs}^2/4+2t\frac{T_{cs}}{2}}{T_{cs}^2/4}\,dt$$

$$= \frac{8A_L^2}{T_{cs}^2}\int_{-T_{cs}/2}^{0}\left(t^2+\frac{\cancel{2}T_{cs}t}{\cancel{2}}+\frac{T_{cs}^2}{4}\right)dt$$

$$= \frac{8A_L^2}{T_{cs}^2}\left(\left.\frac{t^3}{3}\right|_{-T_{cs}/2}^{0} + T_{cs}\left.\frac{t^2}{2}\right|_{-T_{cs}/2}^{0} + \left.\frac{T_{cs}^2}{4}t\right|_{-T_{cs}/2}^{0}\right)$$

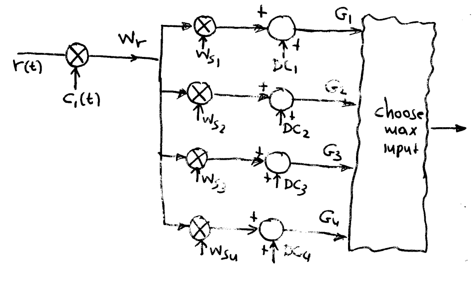
$$= \frac{8A_L^2}{T_{cs}^2}\left(\frac{T_{cs}^3}{3\times8} - \cancel{\frac{T_{cs}^3}{8}} + \cancel{\frac{T_{cs}^3}{8}}\right)$$

$$= \frac{1}{3}\underbrace{T_{cs}}_{12}\cdot A_L^2 = 4\cdot A_L^2$$

$$\therefore\ c_1(t) = +A_1\,\Lambda\left(\frac{2t}{T_{cs}}\right)\Big/\sqrt{E_1} = \frac{\cancel{A_1}}{2\cancel{A_1}}\Lambda\left(\frac{2t}{T_{cs}}\right)$$

$$\therefore\ \boxed{c_1(t) = \frac{1}{2}\,\Lambda\left(\frac{t}{6}\right)}$$

$$\left.\begin{array}{l} \underline{w}_{s_1} = -\sqrt{E_1} = -6\text{mV} \\ \underline{w}_{s_2} = -\sqrt{E_2} = -2\text{mV} \\ \underline{w}_{s_3} = \sqrt{E_3} = 2\text{mV} \\ \underline{w}_{s_4} = \sqrt{E_4} = 6\text{mV} \end{array}\right\}\quad \begin{array}{l} DC_1 = \frac{N_o}{2}\ln(P_1) - \frac{1}{2}E_1 = -20.079\times10^{-6} \\ DC_2 = \frac{N_o}{2}\ln(P_2) - \frac{1}{2}E_2 = -2.98\times10^{-6} \\ DC_3 = \frac{N_o}{2}\ln(P_3) - \frac{1}{2}E_3 = -2.98\times10^{-6} \\ DC_4 = \frac{N_o}{2}\ln(P_4) - \frac{1}{2}E_4 = -20.079\times10^{-6} \end{array}$$

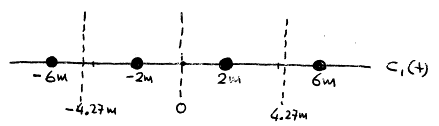


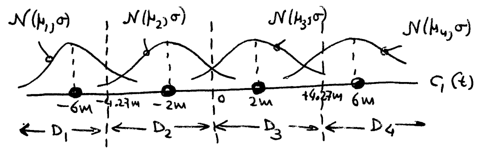
$$G_i = w_r w_{s_i} + DC_i$$

$$G_1 = G_2 \Rightarrow w_r w_{s_1} + DC_1 = w_r w_{s_2} + DC_2 \Rightarrow w_{r_1} = \frac{DC_2 - DC_1}{w_{s_1} - w_{s_2}} = -4.27\text{m}$$

$$G_2 = G_3 \Rightarrow \cdots \Rightarrow w_{r_2} = \frac{DC_3 - DC_2}{w_{s_2} - w_{s_3}} = 0$$

$$G_3 = G_4 \Rightarrow \cdots \Rightarrow w_{r_3} = \frac{DC_4 - DC_3}{w_{s_3} - w_{s_4}} = +4.27\text{m}$$





$\mu_1 = -6\text{m}$

$\mu_2 = -2\text{m}$

$\mu_3 = 2\text{m}$

$\mu_4 = 6\text{m}$

$$\sigma^2 = \frac{N_o}{2}\,2\underbrace{B}_{\frac{1}{2T_{cs}}}\,T_{cs} = \frac{N_o}{2} = 10^{-6} \Rightarrow \boxed{\sigma = 10^{-3}}$$

$$\left.\begin{array}{l} Pr(D_1|H_1) = Pr(D_4|H_4) \\ Pr(D_2|H_2) = Pr(D_3|H_3) \end{array}\right\}\ \text{due to symmetry}$$

$\therefore$

$$P_{e,cs} = 1 - 2\,Pr(D_1|H_1)\,P_1 - 2\,Pr(D_2|H_2)\,P_2$$

$$= 1 - 2\left(1 - T\left(\frac{6-4.27}{10^{-3}}\times10^{-3}\right)\right)\frac{1}{8} - 2\left(1 - T(2) - T(2.27)\right)\frac{3}{8}$$

$$= \frac{2}{8}T(1.73) + \frac{6}{8}T(2) + \frac{6}{8}T(2.27)$$

$= 2/8*0.0418 + 6/8*0.0228 + 6/8*0.0116$
$= 0.0362$