# ISE2
# Discrete Mathematics and Computational Complexity

## Specimen Paper

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1. Compulsory Question

(a) Which (if any) of these statements are propositions? Briefly justify your answers.

(i) $4x = 5$
(ii) $5x + 1 = 5$ if $x = 1$
(iii) $x + y + z = y + 2z$ if $x = z$

[3]

(b) Show that each of these implications is a tautology using an appropriate truth table.

(i) $p \wedge q \rightarrow p$
(ii) $p \rightarrow p \vee q$
(iii) $\neg p \rightarrow (p \rightarrow q)$

[3]

(c) Determine the truth of each of these propositions, where the universe of discourse is the set of integers. Briefly justify your answers.

(i) $\forall n(n^2 \geq n)$
(ii) $\exists n(n^2 = 2)$

[2]

(d) Find the power set of each of the following sets.

(i) $\{a\}$
(ii) $\{a,b\}$
(iii) $\{\emptyset,\{\emptyset\}\}$

[3]

(e) Let $f(n)$ be the function from the set of integers to the set of integers such that $f(n) = n^2 + 1$. What are the domain, codomain, and range of this function?

[3]

(f) Which (if any) of these functions are bijections from **R** to **R**? Briefly justify your answers.

(i) $f(x) = 2x + 1$
(ii) $f(x) = x^2 + 1$
(iii) $f(x) = x^3$

[3]

(g) Find an appropriate big-O expression for each of these recurrence relations.

(i) $f(n) = 2f(n-1)$, with $f(0) = 1$.
(ii) $f(n) = 2f(n-1) + 1$, with $f(0) = 0$.
(iii) $f(n) = 2f(n/3) + n^2$, with $f(0) = 0$.

[3]

Total: 20 Marks

## 2. Logic

Let P($x$) denote the statement "$x$ owns a computer".
Let Q($x$) denote the statement "$x$ can program a computer".
Let R($x$) denote the statement "$x$ has studied computing".
Let S($x$,$y$) denote the statement "$x$ knows $y$".
Let the universe of discourse be the set of all people.

(a) Write the following English statements using symbolic logic.
   (i) Everyone who owns a computer can program a computer.
   (ii) Someone can program a computer, but doesn't own one.
   (iii) Someone who has studied computing can't program a computer.
   (iv) Everyone who can program a computer has studied computing.
   (v) Steven has not studied computing.

[5]

(b) Use symbolic logic to construct a valid argument that results in the conclusion $\neg$ P(Steven), given the premises derived in part (a). At each step of your argument, state the rule of inference used.

[7]

(c) Write the following statements using symbolic logic.
   (i) Someone knows everyone who can program a computer.
   (ii) Everyone knows someone who can program a computer.
   (iii) Everyone who has studied computing knows someone who owns a computer.

[3]

Total: 15 Marks

3. Algorithm Analysis

a) Derive an expression for the number of multiplications performed by the code in Fig. 3.1, in terms of the input value *n*.

```
procedure p(n: integer)
begin
  total := 0
  for i := 1 to n
    for j := 1 to n
      total := total + i*j
  result := total
end
```

Figure 3.1

[2]

b) Hence derive a recurrence relation for the number of multiplications performed by the code in Fig. 3.2, in terms of the input value *n*.

```
procedure q(n: integer)
begin
  if n = 0 then
    result := 1
  else
    result := q(n/2) + p(n)
end
```

Figure 3.2

[3]

c) State the Master Theorem.

[3]

d) Derive a big-O expression for the number of multiplications performed by a call to **q**, in terms of *n*.

[1]

e) Prove that if the recurrence relation $f(n) = a\, f(n/b) + c$ is satisfied, where $a > 1$, then $f(n)$ is $O(n^{\log_b a})$. You need only consider the case where $n = b^k$ for some $k \in \mathbf{Z}^+$.

[6]

Total: 15 Marks

4. Computability

(a) Define what is meant if a problem is said to be *tractable*, and give an example of a tractable problem.

[2]

(b) Define what is meant if a problem is said to be *unsolvable*, and give an example of an unsolvable problem.

[2]

(c) Prove that the problem from part (b) is unsolvable.

[6]

(d) Let the set of ISE2 students be denoted S. Consider a symmetric relation D on S, such that $s_1$ D $s_2$ iff student $s_1$ dislikes student $s_2$.

The "student allocation" problem is defined as:

Can the set of students be partitioned into no more than *n* teams, such that no team contains any two students who dislike each other?

Prove that "student allocation" is at least as hard as *k*-colouring (defined below for convenience).

The k-colouring problem is defined as:

Given a set of nodes V, a set of edges E, and a positive integer *k*, does there exist a function $p : V \rightarrow \{1,2,\ldots,k\}$ such that $\forall v_1 \forall v_2 ( \{v_1, v_2\} \in E \rightarrow p(w_1) \neq p(w_2) )$, where the universe of discourse is the set V?

[5]

Total: 15 Marks.

# Model Answers

Question 1.

(a)

(i) is not a proposition. It is neither true nor false, as $x$ is undefined.
(ii) is a proposition, as it has a definite truth value (false).
(iii) is a proposition, as it has a definite truth value (true).

(b) The truth tables are shown below. In each case, the final column is always true, thus the expression is a tautology.

(i)

| $p$ | $q$ | $p \wedge q$ | $p \wedge q \rightarrow p$ |
|---|---|---|---|
| F | F | F | T |
| F | T | F | T |
| T | F | F | T |
| T | T | T | T |

(ii)

| $p$ | $q$ | $p \vee q$ | $p \rightarrow p \vee q$ |
|---|---|---|---|
| F | F | F | T |
| F | T | T | T |
| T | F | T | T |
| T | T | T | T |

(iii)

| $p$ | $q$ | $\neg p$ | $p \rightarrow q$ | $\neg p \rightarrow (p \rightarrow q)$ |
|---|---|---|---|---|
| F | F | T | T | T |
| F | T | T | T | T |
| T | F | F | F | T |
| T | T | F | T | T |

(c)

(i) If $n$ is negative, clearly $n^2 \geq 0 \geq n$. If $n$ is zero, it is true. If $n$ is positive, since $n$ is an integer, $n \geq 1$, so multiplying both sides by $n$, we obtain $n^2 \geq n$. Thus this proposition is true.
(ii) There is no *integer* $n$ with $n^2 = 2$, so the proposition is false.

(d)

(i) $\{\varnothing,\{a\}\}$
(ii) $\{\varnothing,\{a\},\{b\},\{a,b\}\}$
(iii) $\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\varnothing,\{\varnothing\}\}\}$

(e) Domain: the set of integers, Codomain: the set of integers, Range: $\{n^2 + 1 \mid n \text{ is an integer}\}$.

(f)

(i) Yes. There is an inverse $g(x) = (x\text{-}1)/2$.
(ii) No, as the range is $\{x^2 + 1 \mid x \text{ is real}\}$, which is not the set of reals, so the function is not a surjection. Alternatively, we can simply note that $x = -1$ and $x = +1$ have the same value of $f(x)$, so the function is not an injection.
(iii) Yes. There is an inverse $g(x) = x^{1/3}$.

(g)
(i) $f(n) = 2^n$, which is $O(2^n)$.
(ii) $f(n) = 2^n - 1$, which is $O(2^n)$.
(iii) $f(n)$ is $O(n^2)$ from the Master Theorem.

Question 2.

(a)
(i) $\forall x(\ P(x) \rightarrow Q(x)\ )$
(ii) $\exists x(\ Q(x) \wedge \neg P(x)\ )$
(iii) $\exists x(\ R(x) \wedge \neg Q(x)\ )$
(iv) $\forall x(\ Q(x) \rightarrow R(x)\ )$
(v) $\neg$R(Steven)

(b)
1. Q(Steven) $\rightarrow$ R(Steven) [universal instantiation, from premise (iv)]
2. $\neg$ Q(Steven) [modus tollens, when combined with premise (v)]
3. P(Steven) $\rightarrow$ Q(Steven) [universal instantiation, from premise (i)]
4. $\neg$ P(Steven) [modus tollens, when combined with (3) above]

(c)
(i) $\exists x \forall y(\ Q(y) \rightarrow S(x,y)\ )$
(ii) $\forall x \exists y(\ Q(y) \wedge S(x,y)\ )$
(iii) $\forall x \exists y(\ R(x) \rightarrow S(x,y) \wedge P(y)\ )$

Question 3.

(a) Each outer loop executes $n$ times. Each inner loop executes $n$ times per iteration of the outer loop. There is one multiplication per inner loop iteration. Thus the number of multiplications is $n^2$.

(b) $f(0) = 0$: there are no multiplications in the base case. $f(n) = f(n/2) + n^2$.

(c) [see notes]

(d) $O(n^2)$. [A direct application of the Master Theorem]

(e) [see notes]

Question 4.

(a) [see notes]

(b) [see notes]. The only unsolvable example studied in lectures is the halting problem.

(c) [see notes]

(d) There is a direct correspondence between the two problems. "Dislikes" corresponds to edges, and students correspond to nodes. The relation D is symmetric, so the digraph of the relation is equivalent to the graph to be coloured. The following reduction is appropriate:
(i) Set S = V.
(ii) Set D = $\{(v_1,v_2) \mid \{v_1,v_2\} \in E\}$.
(iii) Set $n = k$.