E3.10
C1.6
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

MATHEMATICS FOR SIGNALS AND SYSTEMS

Monday, 28 April 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	G. Weiss
	Second Marker(s) :	J.C. Allwright

1. In the vector space $V = \mathbb{C}^3$ (with the natural inner product), consider the vector

$$e_1 = \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Find a number $\alpha > 0$ and $e_2, e_3 \in V$ such that $\{e_1, e_2, e_3\}$ is an orthonormal basis in V. [4]

(b) Find a unitary operator $T \in \mathcal{L}(V)$ (expressed as a matrix) such that (using the notation from part (a))

$$Te_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad Te_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hint: is T^{-1} also unitary? [5]

(c) How many eigenvalues of T (from part (b)) are contained in the open unit disk $\mathcal{D}$? Give a brief reasoning. Hint: there is no need to compute these eigenvalues. [2]

(d) Consider the set

$$M = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Give a simple description of the set $M^\perp$. [3]

(e) Let P be the orthogonal projector onto the space $M^{\perp\perp}$, where M is the set defined in part (d). Compute the matrix of P. [4]

(f) For P from part (e), compute P^2, rank P and $\|P\|$. [2]

2. Consider the discrete-time signal $v = (v_k)$ defined by

$$v_k = \frac{\sin 0.01k}{0.01k}, \quad k = 0, 1, 2, 3, \ldots .$$

We denote by $\hat{v}$ the $\mathcal{Z}$-transform of v.

(a) Indicate which of the following statements is true:

$$v \in l^2, \quad v \in l^\infty, \quad v \in l^1, \quad v \in \mathbf{c}_0 .$$ [3]

(b) Sketch the plot of v for k up to about 1600. [3]

(c) Is $\hat{v}$ defined on the unit circle? If yes, is $\int_0^{2\pi} |\hat{v}(e^{i\varphi})|^2 \mathrm{d}\varphi$ finite? Give a brief reasoning. [5]

(d) Consider the digital filter described by the difference equation

$$y_k - 0.8y_{k-1} = 3u_k - 0.5u_{k-1} - 2u_{k-2},$$

where u is the input and y is the output. Is this filter time-invariant? Is this filter stable? Compute the transfer function F of this filter. [4]

(e) We apply the input $u = v$ to the filter in part (d), knowing that the initial conditions of the filter are zero. Indicate which of the following statements about the output y are true:

$$y \in l^2, \quad y \in l^\infty, \quad y \in \mathbf{c}_0, \quad \hat{y} \in H^2(\mathcal{E}), \quad \hat{y} \in H^2(\mathcal{D}).$$

Explain briefly your answer. [5]

3. A nonlinear load is connected to the domestic power grid, so that the voltage on this load is

$$U(t) = 325 \sin 100\pi t$$

(measured in Volts). The current through the load is

$$I(t) = 10 \sin(100\pi t + 0.1) + 2 \sin(300\pi t + 0.3) + 0.1 \sin(700\pi t + 0.7)$$

(in Amps). These formulas hold for all real t.

(a) Sketch the plot of I for t from zero up to about 50 msec. There is no need to give details, such as the precise location of zeros or maximum points, but the two axes should have some values marked. [2]

(b) Compute the RMS vlaue of I, denoted I_{RMS}. (Recall that $I_{RMS}^2 = \frac{1}{T}\int_0^T I(t)^2 \mathrm{d}t$, where T is the period of I.) Hint: use orthogonality to simplify your computation. [4]

(c) Compute the average power absorbed by the load. [4]

(d) Is I a band-limited function (in the sense used in the sampling theorem)? Explain briefly your answer. [3]

(e) Is I an analytic function of t? [3]

(f) Let I_0 be the restriction of I to one period $[t_0, t_0+T]$ (and $I_0(t) = 0$ for all t not in this interval). Can we choose t_0 such that I_0 is continuous? Can we choose t_0 such that I_0 is band-limited? Explain briefly your answer. [4]

4. Consider the following functions defined for $s \in \mathbb{C}_+$:

$$v(s) = \frac{1}{s+2}, \qquad \theta(s) = \frac{s-5}{s+5}, \qquad q(s) = e^{-4s},$$

$$h(s) = \frac{e^{-s}}{s^2+1}, \qquad \psi(s) = \frac{s}{s^2-4}.$$

(a) Which of the functions listed above is in $H^2(\mathbb{C}_+)$? [2]

(b) Which of the functions listed above is in $H^\infty(\mathbb{C}_+)$? For those that you find, compute their norm in $H^\infty(\mathbb{C}_+)$. [3]

(c) Which of the functions you found at part (b), when regarded as a transfer function, determines an isometric operator from input signals in $L^2[0,\infty)$ to output signals in $L^2[0,\infty)$? [3]

(d) Which of the functions listed at the beginning of this question, when extended as an analytic function to the largest possible domain, is analytic on $\mathbb{C}_-$? [2]

(e) Compute the inverse Laplace transforms of all the functions listed at the beginning of this question. [3]

(f) Find $\psi_- \in H^2(\mathbb{C}_-)$ and $\psi_+ \in H^2(\mathbb{C}_+)$ such that

$$\psi = \psi_- + \psi_+ \,. \qquad [3]$$

(g) Compute $J = \int_{-\infty}^{\infty} \psi(i\omega)\overline{v(i\omega)}\mathrm{d}\omega$. Hint: use the orthogonal decomposition from part (f), the inverse Laplace transforms from part (e) and the Paley-Wiener theorem. [4]

5. Consider the linear system described by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -300 \\ 300 & -\beta^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u,$$

$$y = [0 \quad -\beta]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where u is the input signal, x is the state (with two components), y is the output signal and β is a real constant.

(a) For which values of β is the system stable? [4]

(b) Compute the transfer function $\mathbf{G}$ of the system for $\beta = 0.1$ and compute $\|\mathbf{G}\|_\infty$ (its norm in $H^\infty(\mathbb{C}_+)$) with a precision of $\pm 5\%$. [4]

(c) Let T be the input-output map of the system, from inputs in $L^2(-\infty, \infty)$ to outputs in the same space, with $\beta = 0.1$. Compute $\|T\|$ with a precision of $\pm 5\%$ and give a short reasoning. [5]

(d) Let $BL(100)$ be the space of band-limited functions with angular frequencies not higher than 100 rad/sec. Explain briefly why it holds that if $u \in BL(100)$ then also $Tu \in BL(100)$. [2]

(e) Compute the norm of T (defined in part (c)) as an operator from $BL(100)$ to itself, with a precision of $\pm 5\%$. [5]

[END]