IMPERIAL COLLEGE LONDON

E3.10
C1.5
ISE3.7

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

Corrected Copy

MATHEMATICS FOR SIGNALS AND SYSTEMS

Friday, 11 May 10:00 am

Time allowed: 3:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : G. Weiss

Second Marker(s) : J.C. Allwright

1. Consider the matrix

$$T = \begin{bmatrix} 4 & -7 \\ -3 & 8 \\ 0 & 3 \end{bmatrix}.$$

(a) The range of T, denoted Ran T, is (by definition) the subspace of $\mathbb{C}^3$ consisting of all the vectors Tx, where $x \in \mathbb{C}^2$. Determine an orthonormal basis in Ran T. [4]

(b) Let P be the projector from $\mathbb{C}^3$ onto Ran T. Determine an expression for the matrix corresponding to P, in terms of the basis computed in part (a). There is no need to do any numerical computations. [3]

(c) Recall that T^* is the conjugate of the transpose of T. Compute the eigenvalues of T^*T. [3]

(d) Compute the norm of T, which is defined as $\|T\| = \sup_{\|x\|\le 1} \|Tx\|$. (Here, we use the usual Euclidean norm on $\mathbb{C}^2$ and on $\mathbb{C}^3$.) [3]

(e) Find a nonzero vector $z \in \mathbb{C}^3$ such that $T^*z = 0$. [3]

(f) Recall that for every set M in an inner product space, $M^\perp$ denotes the orthogonal complement of M. $\mathbb{C}^3$ is an inner product space with the standard inner product $\langle x, z\rangle = x_1\bar{z}_1 + x_2\bar{z}_2 + x_3\bar{z}_3$.

Prove that if T is a matrix in $\mathbb{C}^{3\times 2}$ and $z \in \mathbb{C}^3$ is such that $T^*z = 0$, then $z \in (\operatorname{Ran} T)^\perp$. [4]

2. Consider the discrete-time signal $v = (v_k)$ defined for $k \in \{0, 1, 2, \ldots\}$ by

$$v_k = \begin{cases} \frac{\cos^2 5k}{k} & \text{if } k \text{ is a prime number,} \\ \frac{1}{k+7} & \text{else.} \end{cases}$$

(The prime numbers are $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$.) We denote by $\hat{v}$ the $\mathcal{Z}$-transform of v.

(a) Indicate which of the following statements is true:

$$v \in l^1, \quad v \in L^2[0,\infty), \quad v \in l^2, \quad v \in l^\infty, \quad v \in \mathbf{c}_0. \qquad [3]$$

(b) Define $f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\hat{v}(z)$. Show that f is the $\mathcal{Z}$-transform of a sequence $w \in l^\infty$. Compute w_0, w_1 and w_2. [3]

(c) Find a point on the unit circle where the series defining $\hat{v}(z)$ is not convergent. [3]

(d) Determine the largest open set in $\mathbb{C}$ where $\hat{v}(z)$ is defined (i.e., the series which defines $\hat{v}(z)$ is convergent). Give a brief reasoning. [4]

(e) Consider the digital filter described by the difference equation

$$4y_k - y_{k-2} = 3u_k - 4u_{k-1} + 5u_{k-2},$$

where u is the input and y is the output. Is this filter time-invariant? Is this filter stable? Compute the transfer function G of this filter. [3]

(f) Let $(g_0, g_1, g_2, \ldots)$ be the impulse response of the filter from part (e). Compute g_0, g_1 and $\lim g_k$. Hint for g_0 and g_1: If G is the transfer function from part (e), write $f(s) = G(1/s)$ as a Taylor series around zero and then compute the first two terms of this Taylor series. [4]

3. (a) Define the Hardy space $\mathcal{H}^2(\mathbb{C}_+)$. (Hint: this space consists of all the analytic functions defined on $\mathbb{C}_+$ which have a certain property.) Define the usual norm $\|\cdot\|_2$ on $\mathcal{H}^2(\mathbb{C}_+)$ and describe an inner product which induces this norm. Is this space complete? [3]

(b) For the functions $f(s) = \frac{1}{s+1}$ and $g(s) = \frac{1}{s-1}$, determine whether they belong to $\mathcal{H}^2(\mathbb{C}_+)$ and give a short explanation for your answer. [3]

(c) Define the Hardy space $\mathcal{H}^\infty(\mathbb{C}_+)$ and its usual norm $\|\cdot\|_\infty$. Determine whether the function θ defined by $\theta(s) = \frac{s-1}{s+1}$ belongs to $\mathcal{H}^\infty(\mathbb{C}_+)$ and give a short reasoning. [3]

(d) Suppose that f is an analytic function defined on an open set Ω that contains all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$, and suppose that $f \in \mathcal{H}^\infty(\mathbb{C}_+)$. Explain very briefly why and how $\|f\|_\infty$ can be computed from the restriction of f to the imaginary axis $i\mathbb{R}$. Compute $\|\theta\|_\infty$, where θ is the function defined in part (c). [3]

(e) Show that for every $n \in \{0, 1, 2, \ldots\}$, the function h_n defined by

$$h_n(s) = \left(\frac{s-1}{s+1}\right)^n \frac{\sqrt{2}}{s+1}$$

belongs to $\mathcal{H}^2(\mathbb{C}_+)$, and compute its norm $\|h_n\|_2$. Hint: Use the results from the previous parts of this question. [4]

(f) Show that the set $\{h_n \mid n = 0, 1, 2, \ldots\}$ is orthonormal in $\mathcal{H}^2(\mathbb{C}_+)$. Hint: When integrating on the imaginary axis, make the change of variable $\frac{1-i\omega}{1+i\omega} = e^{-i\varphi}$. [4]

4. A nonlinear load is connected to the utility grid and the voltage on this load is

$$U(t) = 325 \sin 100\pi t$$

(measured in Volts). The current through the load is

$$I(t) = 9 \sin(100\pi t + 0.1) + 2 \sin(300\pi t + 0.3)$$

(in Amps). These formulas hold for all real t.

(a) Sketch the plot of I for t from zero up to about 50 msec. There is no need to give details, such as the precise location of zeros or maximum points, but the two axes should have some values marked. [2]

(b) Compute the RMS value of I, denoted I_{RMS}. (Recall that $I_{RMS}^2 = \frac{1}{T}\int_0^T I(t)^2 \mathrm{d}t$, where T is the period of I.) Hint: use orthogonality to simplify your computation. [4]

(c) Compute the average power absorbed by the load. [4]

(d) Is I a band-limited function (in the sense used in the sampling theorem)? Is I analytic? Explain briefly your answers. [3]

(e) Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \frac{\sin \varepsilon t}{\varepsilon t},$$

where $\varepsilon > 0$. Sketch the graph of φ and compute its Fourier transform (your answer may contain a multiplicative constant of undetermined value). Is φ a band-limited function? [3]

(f) With φ as in part (e), prove that the function J defined by

$$J(t) = I(t)\varphi(t)$$

is band-limited. Hint: First show that for every $\gamma \in \mathbb{R}$, the function $g_\gamma(t) = e^{i\gamma t}\varphi(t)$ is band-limited. Then express $\cos \gamma t$ and $\sin \gamma t$ using $e^{i\gamma t}$ and $e^{-i\gamma t}$. [4]

5. Consider the linear system described by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & -20 \\ 20 & -d^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ d \end{bmatrix} u,$$

$$y = [0 \;\; 1]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where u is the input signal, z is the state (with two components), y is the output signal and d is a real constant.

(a) For which values of d is the system stable? [4]

(b) Compute the transfer function $\mathbf{G}$ of the system for $d = 0.1$ and compute $\|\mathbf{G}\|_\infty$ (its norm in $H^\infty(\mathbb{C}_+)$) with a precision of $\pm 5\%$. [4]

(c) Let T be the input-output map of the system, from inputs in $L^2(-\infty, \infty)$ to outputs in the same space, with $d = 0.1$. Compute $\|T\|$ with a precision of $\pm 5\%$ and give a short reasoning. [5]

(d) Let $BL(10)$ be the space of band-limited functions with angular frequencies not higher than 10 rad/sec. Explain briefly why it holds that if $u \in BL(10)$ then also $Tu \in BL(10)$. [2]

(e) Compute the norm of T (defined in part (c)) as an operator from $BL(10)$ to itself, with a precision of $\pm 5\%$. [5]

[END]

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None of this paper is bookwork and all examples are new to the class.

Mathematics for Signals and Systems

Exam of May 2007

SOLUTIONS

Question 1 (a) We normalize the first column in T: $e_1 = \alpha \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$,

where $\alpha = \frac{1}{\sqrt{4^2+3^2}} = \frac{1}{\sqrt{25}} = 0.2$, so that

$\|e_1\| = 1$. Now we take $e_2 = \beta\left(\begin{bmatrix} -7 \\ 8 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}\right)$.

To make sure that $\langle e_1, e_2 \rangle = 0$, we have to take

$$\gamma = -\left\langle \begin{bmatrix} -7 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\rangle \Big/ \left\langle \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\rangle = 2.08,$$

so that

$$e_2 = \beta \begin{bmatrix} -7 + 2.08 \cdot 4 \\ 8 - 2.08 \cdot 3 \\ 3 \end{bmatrix} = \beta \begin{bmatrix} 1.32 \\ 1.76 \\ 3 \end{bmatrix}.$$

For $\|e_2\| = 1$ we take $\beta = 1 \Big/ \sqrt{(1.32)^2 + (1.76)^2 + 3^2}$,

$\beta = \frac{1}{\sqrt{13.84}} = 0.2688\ldots$, so that finally

$$e_2 = \begin{bmatrix} 0.354818\ldots \\ 0.473091\ldots \\ 0.806405\ldots \end{bmatrix}.$$

Now $\{e_1, e_2\}$ is an orthonormal basis in Ran T. [4]

(b) $$Px = e_1 \langle x, e_1 \rangle + e_2 \langle x, e_2 \rangle$$

$$= (e_1 e_1^* + e_2 e_2^*) x,$$

so that

$$P = e_1 e_1^* + e_2 e_2^* . \quad [3]$$

(c) $$T^* T = \begin{bmatrix} 25 & -52 \\ -52 & 122 \end{bmatrix},$$ the characteristic polynomial of this matrix is

$$\det(\lambda I - T^* T) = \lambda^2 - 147\lambda + 346,$$

so that the eigenvalues of T^*T are

$$\lambda_{1,2} = \frac{147 \pm \sqrt{20{,}225}}{2}$$

$$\lambda_1 = 2.39269\ldots \qquad \lambda_2 = 144.6073\ldots \quad [3]$$

(d) $$\|T\|^2 = \max\{\lambda_1, \lambda_2\} = 144.6073\ldots,$$

so that $$\|T\| = 12.02528\ldots \quad [3]$$

(e) We want to find $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ such that $T^* z = 0$, i.e.,

$$\begin{bmatrix} 4 & -3 & 0 \\ -7 & 8 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $z \neq 0$.

Take $z_3 = 1$, then we obtain

$$\underbrace{\begin{bmatrix} 4 & -3 \\ -7 & 8 \end{bmatrix}}_{T_0} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

This can be solved, because $\det T_0 =$ $= 11 \neq 0$. The solution is $z_1 = -\frac{9}{11}$, $z_2 = -\frac{12}{11}$, so that $z = \begin{bmatrix} \frac{-9}{11} & \frac{-12}{11} & 1 \end{bmatrix}^T$. [3]

(f) We use the standard inner product

$$\langle x, z \rangle = x_1 \bar{z}_1 + x_2 \bar{z}_2 + x_3 \bar{z}_3 .$$

Assume that $T \in \mathbb{C}^{3\times 2}$, $z \in \mathbb{C}^3$, $T^* z = 0$ (we have seen an example of such a T and z in the answer to part (e)).

We want to show that $z \in (\operatorname{Ran} T)^\perp$. Any element $m \in \operatorname{Ran} T$ is of the form $m = Tx$, for some $x \in \mathbb{C}^2$. Then

$\langle m, z \rangle = m^* z = x^* T^* z = 0.$

Thus, z is orthogonal to every $m \in \operatorname{Ran} T$.

[4]

$\boxed{\text{Question 2}}$ (a) <u>v is not in ℓ^1</u>. Indeed,

$$\sum_{k\in\mathbb{N}} |v_k| \geq \sum_{\substack{k=2n\\ n\geq 2}} |v_k| = \sum_{n=2}^{\infty} \frac{1}{2n+7} = \infty .$$

We have <u>$v \in \ell^2$</u>. Indeed, for $k \geq 1$,

$|v_k| \leq \frac{1}{k}$ (no matter if k is prime or not)

and $(\frac{1}{k}) \in \ell^2$. Hence, <u>$v \in c_0 \subset \ell^\infty$</u>. Clearly we do not have $v \in L^2[0,\infty)$, this is nonsense.

[3]

(b) $\hat{v}(z) = v_0 + v_1 z^{-1} + v_2 z^{-2} + v_3 z^{-3} + \dots$

$$f(z) = \frac{d}{dz}\hat{v}(z) = -v_1 z^{-2} - 2v_2 z^{-3} - 3v_3 z^{-4} - \dots$$

Hence, $f = \hat{w}$, where $w_k = -(k-1)\, v_{k-1}$ for $k \geq 2$, while <u>$w_0 = w_1 = 0$</u>. We have $w_2 = -v_1$ so that <u>$w_2 = -\frac{1}{1+7} = -1/8$</u>. We have seen at our answer to part (a) that $|v_k| \leq \frac{1}{k}$ for $k \geq 1$, hence $|w_k| \leq (k-1)/k \leq 1$ for $k \geq 1$. Since $w_0 = 0$, we have $|w_k| \leq 1$ for all k. [3]

(c) We have seen in our answer to part (a) that $\sum_{k=0}^{\infty} v_k = \sum_{k=0}^{\infty} |v_k| = \infty$. Hence, the series for $\hat{v}(z)$ is <u>not convergent for $z=1$.</u>

[3]

(d) Since $v \in l^2$, $\hat{v}$ is defined on $\mathcal{E}$ (the exterior of the unit disk). We have seen in our answer to part (c) that there are points on the unit circle where $\hat{v}$ is not defined. Hence, $\hat{v}$ cannot be defined on the exterior of a smaller disk. We know that a $\mathcal{Z}$ transform is always defined on the exterior of some disk. Thus, the largest open set on which $\hat{v}$ is defined is $\mathcal{E}$. [4]

(e) The filter is obviously time-invariant. Its transfer function is

$$G(z) = \frac{3 - 4z^{-1} + 5z^{-2}}{4 - z^{-2}} = \frac{3z^2 - 4z + 5}{4z^2 - 1}.$$

The poles of G are $p_1 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, which are in the open unit disk. Hence, this filter is stable. [3]

(f) $$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3} + \dots$$

holds for all $z \in \mathbb{C}$ with $|z| > \frac{1}{2}$ (because the poles of G are at $\pm\frac{1}{2}$). Thus, $f(s) = G\left(\frac{1}{s}\right) = g_0 + g_1 s + g_2 s^2 + g_3 s^3 + \dots$ for all $s \in \mathbb{C}$ with $|s| < 2$. We have $g_0 = f(0) = G(\infty)$, hence $g_0 = 3/4$. Similarly, $g_1 = f'(0) = -1$. [4]

Question 3 (a) $\mathcal{H}^2(\mathbb{C}_+)$ consists of all the analytic functions f defined on $\mathbb{C}_+$ which have the property

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |f(\alpha+i\omega)|^2 d\omega < \infty.$$

The above expression is, by definition, $\|f\|_2^2$. If we denote by $f^*(i\omega)$ the boundary values of f at points on the imaginary axis (these are computed as $\lim_{\substack{\alpha\to 0\\ \alpha>0}} f(\alpha+i\omega)$) then we can define an inner product on $\mathcal{H}^2(\mathbb{C}_+)$ by

$$<f,g> = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(i\omega)\overline{g^*(i\omega)}d\omega.$$

The norm on $\mathcal{H}^2(\mathbb{C}_+)$ is induced by this inner product. This space is complete. [3]

(b) If $f(s) = \frac{1}{s+1}$, then $f \in \mathcal{H}^2(\mathbb{C}_+)$.

Indeed, for each $\alpha > 0$,

$$\int_{-\infty}^{\infty} \left|\frac{1}{\alpha+i\omega+1}\right|^2 d\omega < \int_{-\infty}^{\infty} \frac{1}{\omega^2+1}\, d\omega < \infty,$$

and the bound $\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2+1}$ is independent of α.

If $g(s) = \frac{1}{s-1}$, then g has a pole at 1, hence it is not analytic on $\mathbb{C}_+$, hence g is not in $\mathcal{H}^2(\mathbb{C}_+)$. [3]

(c) $\mathcal{H}^\infty(\mathbb{C}_+)$ consists of all the analytic functions g defined on $\mathbb{C}_+$ for which

$$\sup_{s \in \mathbb{C}_+} |g(s)| < \infty .$$

The above expression is, by definition, $\|g\|_\infty$. If $\theta(s) = \frac{s-1}{s+1}$, then clearly θ is analytic on $\mathbb{C}_+$ and continuous on the closed right half-plane clos $\mathbb{C}_+$. The limit of θ at ∞ is 1, so that θ is bounded on $\mathbb{C}_+$. Thus, $\theta \in \mathcal{H}^\infty(\mathbb{C}_+)$. [3]

(d) If f is bounded on $\mathbb{C}_+$ (and analytic) then by the maximum modulus theorem, the supremum of $|f(s)|$ (when $s \in \mathbb{C}_+$) equals the supremum of $|f^*(i\omega)|$ (here, f^* is as in the answer to part (a)). If f is analytic on an open set containing the closed right half-plane, then f^* is simply the restriction of f to $i\mathbb{R}$. Hence, in this case,

$$\|f\|_\infty = \sup_{\omega \in \mathbb{R}} |f(i\omega)| .$$

For example, this is true for $\theta(s) = \frac{s-1}{s+1}$ in place of s. Since $|\theta(i\omega)| = \frac{|i\omega - 1|}{|i\omega+1|} = \frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+1}} = 1$, we obtain that $\|\theta\|_\infty = 1$. [3]

(e) If $h_n(s) = \left(\frac{s-1}{s+1}\right)^n \frac{\sqrt{2}}{s+1}$, then for every $s \in \mathbb{C}_+$ we have

$$|h_n(s)| \leqslant \left|\frac{s-1}{s+1}\right|^n \frac{\sqrt{2}}{|s+1|} \leqslant \frac{\sqrt{2}}{|s+1|}.$$

We have seen in part (b) that $\frac{1}{s+1}$ is in $\mathcal{H}^2(\mathbb{C}_+)$, hence $h_n \in \mathcal{H}^2(\mathbb{C}_+)$. We have

$$\|h_n\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |h_n(i\omega)|^2 d\omega = \frac{2}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{|i\omega+1|}.$$

Here, we have used that $\left|\frac{i\omega-1}{i\omega+1}\right| = 1$. Hence

$$\|h_n\|_2^2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2+1} = \frac{1}{\pi} \operatorname{arctg} \omega \Big|_{-\infty}^{\infty} = 1. \text{ [4]}$$

(f) We only have to show that $\langle h_n, h_m\rangle = 0$ for $n \neq m$. We have (for $n \neq m$)

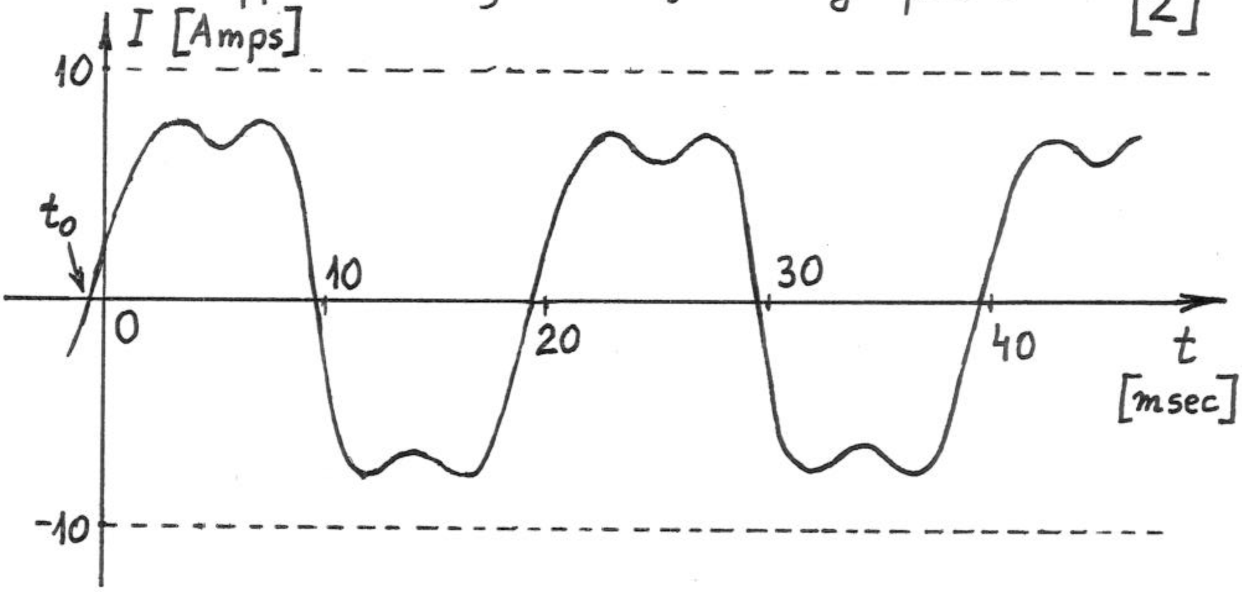
$$\langle h_n, h_m\rangle = \frac{2}{2\pi}\int_{-\infty}^{\infty} \left(\frac{i\omega-1}{i\omega+1}\right)^n \left(\frac{-i\omega-1}{-i\omega+1}\right)^m \frac{d\omega}{(i\omega+1)(-i\omega+1)}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{i\omega-1}{i\omega+1}\right)^{n-m} \frac{d\omega}{\omega^2+1}.$$

We make the change of variable $\frac{1-i\omega}{1+i\omega} = e^{-i\varphi}$, so that φ varies from $-\pi$ to π.
It is easy to see that $\frac{2}{(1+i\omega)^2} d\omega = e^{-i\varphi} d\varphi$, hence $\frac{d\omega}{\omega^2+1} = \frac{d\varphi}{2}$. Thus, $\langle h_n, h_m\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(-e^{i\varphi}\right)^{n-m} \frac{d\varphi}{2}$

$$= \frac{(-1)^{n-m}}{2\pi}\int_{-\pi}^{\pi} e^{i\varphi(n-m)} d\varphi = 0. \qquad \text{[4]}$$

Question 4 (a) The period of I (and also of U) is $2\pi/100\pi = \frac{1}{50}$ sec $= 20$ msec. The fundamental component (of 50 Hz) has amplitude 9, and there is also a third harmonic (of 150 Hz) that has amplitude 2. Both components cross zero simultaneously at $t_0 = -1/1000\pi$. Sketching these two components of I and adding them, we obtain approximately the following plot: [2]



(b) On $L^2[0,T]$ (where $T = 20$ msec) we define the inner product $\langle f, g\rangle = \frac{1}{T}\int_0^T f(t)\overline{g(t)}\,dt$, and we put $\|f\|^2 = \langle f, f\rangle$. Then $I_{RMS} = \|I\|$. Denote (for $k = 1, 3$)

$$e_k(t) = \sin k\,100\pi(t - t_0), \quad t_0 = \frac{-1}{1000\pi}.$$

Then $\langle e_1, e_3\rangle = 0$ and $\|e_1\|^2 = \|e_3\|^2 = \frac{1}{2}$.

Hence $\|I\|^2 = \|9e_1 + 2e_3\|^2$,

$$\|I\|^2 = \|9e_1\|^2 + 2\langle 9e_1, 2e_3\rangle + \|2e_3\|^2$$

$$= 81\,\|e_1\|^2 + 2\cdot 9\cdot 2\,\langle e_1, e_3\rangle + 4\,\|e_3\|^2$$

$$= 81\cdot\frac{1}{2} + 4\cdot\frac{1}{2} = 40.5 + 2 = 42.5$$

so that $I_{RMS} = \sqrt{42.5} \approx 6.519\ldots$ (Amps) [4]

(c) $P = \langle U, I\rangle = \langle U, 9e_1\rangle$ (because U is orthogonal to e_3)

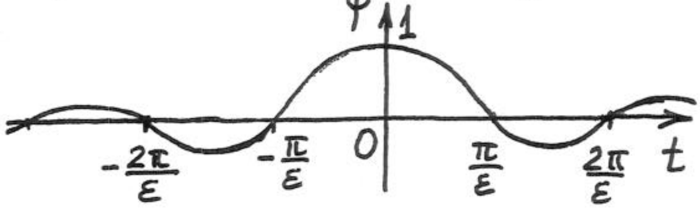
$$= 325\cdot 9\,\langle \sin 100\pi t, \sin(100\pi t + 0.1)\rangle$$

$$= 2925\cdot\frac{1}{2}\cdot\cos 0.1 \approx 1455.24\ldots \text{ (Watts)}$$

[4]

(d) I is not in $BL(\omega)$ for any $\omega > 0$, because it is not in $L^2(-\infty,\infty)$. It is clearly analytic (we could take $t\in\mathbb{C}$). [3]

(e) $\varphi(t) = \dfrac{\sin\varepsilon t}{\varepsilon t}$



$$(\mathcal{F}\varphi)(i\omega) = \begin{cases} k & \text{for } |\omega| \leq \varepsilon, \\ 0 & \text{else}. \end{cases}$$

(Here, $k = \dfrac{\pi}{\varepsilon}$ but this is not important.)

Hence, $\varphi \in BL(\varepsilon)$. [3]

(f) If $g_\gamma(t) = e^{i\gamma t}\varphi(t)$, then $(\mathcal{F}g_\gamma)(i\omega) =$

$$= \begin{cases} k & \text{for } |\omega - \gamma| \leq \varepsilon, \\ 0 & \text{else}, \end{cases}$$ so that $g_\gamma \in BL(|\gamma| + \varepsilon)$.

We can write I as a linear combination of $e^{i\gamma t}, e^{-i\gamma t}, e^{3i\gamma t}, e^{-3i\gamma t}$, where $\gamma = 100\pi$. [4]

$\boxed{\text{Question 5}}$ (a) Denoting the 2×2 matrix by A, the characteristic polynomial of A is

$$p(s) = \det(sI - A) = s^2 + d^2 s + 400.$$

A is stable iff the coefficients of p are positive. Thus, the system is stable iff $d \neq 0$. [4]

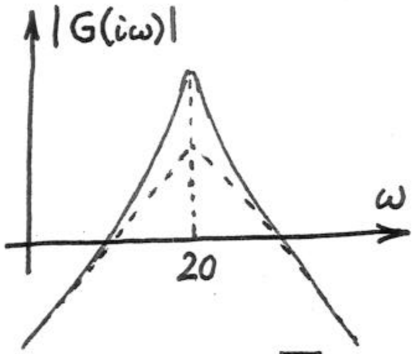
(b) We have $$(sI-A)^{-1} = \frac{1}{p(s)} \begin{bmatrix} s+d^2 & -20 \\ 20 & s \end{bmatrix},$$

whence

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} (sI-A)^{-1} \begin{bmatrix} 0 \\ d \end{bmatrix} = \begin{bmatrix} 20 & s \end{bmatrix} \frac{1}{p(s)} \begin{bmatrix} 0 \\ d \end{bmatrix}$$

$$= \frac{sd}{p(s)} = \frac{0.1 s}{s^2 + 0.01 s + 400}.$$

The Bode plot of G (shown below) reveals that



$|G|$ has a peak at $\omega = 20$. To obtain the peak value, we substitute $s = 20i$ into $G(s)$, which gives $$G(20i) = \frac{2i}{0.2i} = 10.$$

Thus, $\|G\|_\infty = 10$ (exactly). [4]

(c) We have $Tu = \mathcal{F}^{-1} G \mathcal{F}$, for all $u \in L^2(-\infty, \infty)$. (Indeed, for $u \in L^2[0,\infty)$ this follows from the Fourés-Segal theorem, for $u \in L^2[\alpha, \infty)$ it follows by shifting u to the right by $-\alpha$ (assuming $\alpha < 0$), and for $u \in L^2(-\infty,\infty)$ we can approximate u by functions in $L^2[\alpha,\infty)$ with $\alpha < 0$ and $|\alpha|$ very large.) Since $\mathcal{F}$ is unitary, it follows that $\|T\|$ is the norm of the multiplication operator by G on

$L^2(i\mathbb{R})$. It is easy to see that the norm of this multiplication operator is $\sup_{\omega\in\mathbb{R}} |G(i\omega)| = \|G\|_\infty = 10$. Thus, $\|T\| = 10$ (exactly). [5]

(d) From $Tu = \mathcal{F}^{-1} G \mathcal{F} u$ we see that if $(\mathcal{F}u)(i\omega) = 0$ for $|\omega| > 10$, then also $(\mathcal{F}Tu)(i\omega) = 0$ for $|\omega| > 10$. Moreover, if $u \in L^2(-\infty,\infty)$, then $\mathcal{F}u \in L^2(i\mathbb{R})$ (by Paley-Wiener) and, since G is bounded on $i\mathbb{R}$, it follows that also $\mathcal{F}Tu = G\mathcal{F}u \in L^2(i\mathbb{R})$. Hence, $Tu \in BL(10)$. [2]

(e) For $\omega \in (0,10)$, $|G(i\omega)|$ is an increasing function (see the Bode plot at our answer to part (b)). Thus, the maximal gain on the relevant frequency range is attained at $\omega = 10$. We have

$$G(10i) = \frac{i}{-100 + 0.1i + 400} = \frac{i}{300 + 0.1i}$$

$$|G(10i)| \approx \frac{1}{300} = 0.003333\ldots .$$

Thus, the norm of T restricted to $BL(10)$ is $\approx 0.00333\ldots$, much less than its norm on $L^2(-\infty,\infty)$. [5]