

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING  
EXAMINATIONS 2006

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Tuesday, 25 April 10:00 am

Time allowed: 3:00 hours

There are **FIVE** questions on this paper.

Answer **THREE** questions.

*All questions carry equal marks*

**Corrected Copy**

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible

First Marker(s) : G. Weiss

Second Marker(s) : J.C. Allwright

1. Consider the space  $\mathcal{H} = \mathbb{C}^{3 \times 3}$  of matrices with three rows and three columns. We define an inner product on  $\mathcal{H}$  by  $\langle A, B \rangle = \text{trace} B^* A$ , where  $B^*$  is the complex conjugate of the transpose of  $B$ , and we define the corresponding norm on  $\mathcal{H}$  by  $\|A\|_{\mathcal{H}}^2 = \langle A, A \rangle$ .

- (a) What is the dimension of  $\mathcal{H}$ ? [1]  
 (b) In the sequel we denote

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Compute  $\|S\|_{\mathcal{H}}$ . Compute also the norm  $\|S\|$  when  $S$  is regarded as an operator from  $\mathbb{C}^3$  to  $\mathbb{C}^3$ . (Hint: be careful, the norm of  $S$  as an operator is not the same as  $\|S\|_{\mathcal{H}}$ .) [3]

- (c) We say that a matrix  $A \in \mathcal{H}$  is *S-invariant* if  $AS = SA$ . In the sequel we denote by  $\mathcal{F}$  the set of all the *S-invariant* matrices in  $\mathcal{H}$ . Show that  $\mathcal{F}$  is actually a subspace of  $\mathcal{H}$ . [2]  
 (d) Determine the dimension of  $\mathcal{F}$ . [2]  
 (e) Show that if  $A, B \in \mathcal{F}$ , then also  $AB \in \mathcal{F}$  and  $AB = BA$ . [3]  
 (f) Find an orthonormal basis in  $\mathcal{F}$ . [3]  
 (g) Show that if  $A \in \mathcal{F}$ , then  $A$  has only one eigenvalue. [2]  
 (h) Find a non-zero vector  $x \in \mathbb{C}^3$  such that for every *S-invariant* matrix  $A$ ,  $x$  is an eigenvector of  $A$ . [3]  
 (i) Explicitly describe all the matrices  $A \in \mathcal{F}$  for which  $A^* \in \mathcal{F}$ . [1]

2. For  $1 \leq p < \infty$ , we denote by  $l^p$  the space of all sequences  $u$  indexed by  $k \in \{0, 1, 2, 3, \dots\}$  for which  $\sum_{k=0}^{\infty} |u_k|^p < \infty$ . For such sequences  $u$ , we use the notation  $\|u\|_p = (\sum_{k=0}^{\infty} |u_k|^p)^{\frac{1}{p}}$ . We denote by  $l^\infty$  the space of all bounded sequences, and let  $\|u\|_\infty = \sup |u_k|$ .

A linear discrete-time system with input  $u$  and output  $y$  is defined by the formula

$$y_k = u_k + 2u_{k-1}, \quad k = 0, 1, 2, 3, \dots$$

The signals  $u$  and  $y$  are defined for integer times  $k \geq 0$  and we consider  $u_{-1} = 0$  (this occurs for  $k = 0$  in the above formula).

- (a) In the sequel, we denote by  $T$  the input-output operator of the above system. Is  $T$  time-invariant? Determine its impulse response  $g$  and compute its transfer function  $\mathbf{G}$ . [2]
- (b) With the notation from part (a), is  $\mathbf{G}$  stable? Is it strictly proper? Is this a finite impulse response (FIR) system? What is the DC gain of this system? [2]
- (c) Show that for every  $p$  ( $1 \leq p \leq \infty$ ), if  $u \in l^p$  and  $y = Tu$ , then also  $y \in l^p$  and

$$\|y\|_p \leq 3\|u\|_p. \quad [3]$$

- (d) Show that if  $u \in l^2$  and  $y = Tu$ , then

$$\|y\|_2 \geq \|u\|_2. \quad [5]$$

- (e) Let  $\hat{y}$  denote the  $\mathcal{Z}$ -transform of  $y$ . Show that if  $u \in l^2$  and  $y = Tu$ , then  $\hat{y}(-2) = 0$ . Show that the operator  $T \in \mathcal{L}(l^2, l^2)$  is not onto. [4]
- (f) Show that there exist operators  $L \in \mathcal{L}(l^2, l^2)$  such that  $LT = I$  (the identity on  $l^2$ ). Show that  $L$  can be chosen such that  $\|L\| \leq 1$ . Show that  $L$  cannot be chosen such that it is time-invariant. Hint: Denote  $V = \{Tu | u \in l^2\}$  (this is the range space of  $T$ ). Define  $L$  on  $V$  using  $\mathcal{Z}$ -transforms, while  $L$  on  $V^\perp$  can be chosen in an arbitrary way. [4]

3. In this question,  $\mathbf{S}_\tau$  denotes the right shift operator by  $\tau$  on  $L^2[0, \infty)$ .
- (a) Define the natural inner product and the corresponding norm on the space  $L^2[0, \infty)$ . For  $s \in \mathbf{C}_+$  and  $\varphi \in L^2[0, \infty)$  defined by  $\varphi(t) = e^{-st}$ , compute  $\|\varphi\|_2$ . [3]
- (b) Let  $u \in L^2[0, \infty)$  and let  $\hat{u}$  denote its Laplace transform. Show that

$$|\hat{u}(s)| \leq \frac{\|u\|_2}{\sqrt{2\operatorname{Re} s}} \quad \text{for all } s \in \mathbf{C}_+.$$

Hint: use the result about  $\|\varphi\|_2$  from part (a) and the Cauchy-Schwarz inequality. [3]

- (c) Let  $y \in L^1[0, \infty)$ , let  $\hat{y}$  denote its Laplace transform and, as usual, denote  $\|y\|_1 = \int_0^\infty |y(t)| dt$ . Show that

$$|\hat{y}(s)| \leq \|y\|_1 \quad \text{for all } s \in \mathbf{C}_+. \quad [3]$$

- (d) In the sequel, consider  $f$  to be the characteristic function of the interval  $[0, 4]$  and  $g(t) = e^{5t}$ ,  $t \geq 0$ . (Thus,  $f(t) = 1$  for  $t \in [0, 4]$  and  $f(t) = 0$  for  $t > 4$ .) We also define  $m = fg$ . Compute the Laplace transforms  $\hat{f}$ ,  $\hat{g}$  and  $\hat{m}$ . [2]
- (e) Which of  $\hat{f}$ ,  $\hat{g}$  and  $\hat{m}$  is rational? Which of these functions belongs to  $H^\infty(\mathbf{C}_+)$ ? Determine the poles of  $\hat{f}$ ,  $\hat{g}$  and  $\hat{m}$ . Hint: for the question concerning  $H^\infty(\mathbf{C}_+)$ , you may use the result from part (c). [3]
- (f) Define  $h = \mathbf{S}_4 m$ , i.e.,  $h$  is obtained by delaying  $m$  by 4 time units. Compute its Laplace transform  $\hat{h}$ , its norm  $\|\hat{h}\|_2$  and the inner product  $\langle m, h \rangle$ . [3]
- (g) Compute

$$\|\hat{m}\|_2, \quad \|\hat{h}\|_2 \quad \text{and} \quad \langle \hat{m}, \hat{h} \rangle,$$

where the norms and the scalar products correspond to the Hardy space  $H^2(\mathbf{C}_+)$ . [3]

4. Let  $L \in \mathbb{C}^{2 \times 2}$ ,  $H \in \mathbb{C}^{2 \times 1}$  and consider the system described by

$$\begin{aligned}\dot{p}(t) &= Lq(t), \\ \dot{q}(t) &= -L^*p(t) + Hu(t), \\ y(t) &= H^*q(t),\end{aligned}$$

where  $u$  is the scalar input signal,  $p(t), q(t) \in \mathbb{C}^2$  and  $y$  is the scalar output signal. (As usual,  $L^*$  denotes the complex conjugate of the transpose of  $L$  and a dot denotes differentiation with respect to the time.) We define the “energy in the system” by

$$E(t) = \frac{1}{2} \left( \|p(t)\|^2 + \|q(t)\|^2 \right).$$

(a) Write the equations of the system in the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where  $x$  is the state of the system and  $A^* = -A$ ,  $C = B^*$ . [2]

The notation  $A, B, C$  and  $x(t)$  will be used also in the sequel.

- (b) Prove that all the eigenvalues of  $A$  are on the imaginary axis. Is this system stable? Hint:  $iA$  is self-adjoint. [3]
- (c) Show that  $\dot{E}(t) = \operatorname{Re} u(t)\overline{y(t)}$ . [3]
- (d) Express the transfer function  $\mathbf{G}$  of this system, in terms of the matrices  $L, H$ , and also in terms of  $A, B, C$ . [3]
- (e) Recall that if  $u = 0$ , then  $x(t) = e^{tA}x(0)$ . Show that  $e^{tA}$  is a unitary operator on  $\mathbb{C}^4$  (for every  $t \geq 0$ ). Hint: use part (c). [3]
- (f) For  $\mathbf{G}$  as in part (d), find a function  $k : \mathbb{C}_+ \rightarrow \mathbb{R}$  such that

$$\mathbf{G}(s) + \mathbf{G}(s)^* = k(s)C(sI - A)^{-1}(\bar{s}I - A^*)^{-1}B,$$

for all  $s \in \mathbb{C}_+$ . Hint: for every  $s, \beta$  that are not eigenvalues of  $A$ ,  $(sI - A)^{-1} - (\beta I - A)^{-1} = (\beta - s)(sI - A)^{-1}(\beta I - A)^{-1}$ . [3]

- (g) Assume that  $H \neq 0$ . Using the result from part (f), show that the transfer function  $\mathbf{G}$  is “strictly positive”, which means that

$$\operatorname{Re} \mathbf{G}(s) > 0 \quad \text{for all } s \in \mathbb{C}_+. \quad [3]$$

5. Consider the system with input  $u$  and output  $y$  described by the differential equation

$$\ddot{y} + 0.2\dot{y} + 100y = 2\dot{u} - u.$$

We denote its transfer function by  $\mathbf{G}$ .

- (a) Compute  $\mathbf{G}$  and determine if it is stable. [2]
- (b) Sketch the Bode amplitude plot of  $\mathbf{G}$  and estimate  $\|\mathbf{G}\|_\infty$  with a precision of  $\pm 20\%$ . [3]
- (c) Define the space  $BL(\omega_b)$  of band-limited functions with angular frequencies not higher than  $\omega_b$ . [3]
- (d) Find an orthonormal basis in  $BL(\omega_b)$ . [3]
- (e) Suppose that  $u \in BL(3)$  and  $v(t) = u(t) \cos 50t$  for all  $t \in \mathbb{R}$ . Determine if  $v$  is band-limited and, if yes, what is its band-limit (i.e., the smallest  $\omega_b > 0$  such that  $v \in BL(\omega_b)$ ). [3]
- (f) Show that  $u$  and  $v$  from part (e) are orthogonal to each other. [3]
- (g) Suppose that  $u$  from part (e) is the input signal of the system considered earlier, and  $y$  is the corresponding output function (defined for all  $t \in \mathbb{R}$ ). Show that  $y \in BL(3)$  and  $\|y\| \leq \|u\|$  (these norms are computed in  $L^2(\mathbb{R})$ ). Hint: you will need the Bode plot from part (b) to answer this part. [3]

[ END ]

# Mathematics for Signals & Systems

Exam of May 2006

## SOLUTIONS

**Question 1** Note that if  $A = [A_{ij}]$ ,  
 $i, j = 1, 2, 3$ , then  $\|A\|_{\mathcal{H}}^2 = \sum_{i, j \in \{1, 2, 3\}} |A_{ij}|^2$ .

(a)  $\dim \mathcal{H} = 9$ .

(b)  $\|S\|_{\mathcal{H}} = \sqrt{2}$ .  $S^*S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,

$\sigma(S^*S) = \{0, 1\}$ , hence  $\|S\| = 1$ .

(c) If  $AS = SA$  and  $BS = SB$ , then clearly  $(A+B)S = S(A+B)$  and  $(\lambda A)S = S(\lambda A)$  for every  $\lambda \in \mathbb{C}$ . Thus,  $\mathcal{F}$  is a subspace of  $\mathcal{H}$ .

(d)  $AS = \begin{bmatrix} A_{12} & A_{13} & 0 \\ A_{22} & A_{23} & 0 \\ A_{32} & A_{33} & 0 \end{bmatrix}$ ,  $SA = \begin{bmatrix} 0 & 0 & 0 \\ A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ ,

if the above are equal (i.e.,  $A \in \mathcal{F}$ ) then  $A$  must have the structure

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}, \text{ with } \alpha, \beta, \gamma \in \mathbb{C}.$$

From here it is clear that  $\dim \mathcal{F} = 3$ .

(e) If  $A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}$  and  $B = \begin{bmatrix} \delta & 0 & 0 \\ \epsilon & \delta & 0 \\ \eta & \epsilon & \delta \end{bmatrix}$ , then

$$AB = BA = \begin{bmatrix} \alpha\delta & 0 & 0 \\ \beta\delta + \alpha\epsilon & \alpha\delta & 0 \\ \gamma\delta + \beta\epsilon + \alpha\eta & \beta\delta + \alpha\epsilon & \alpha\delta \end{bmatrix}.$$

(f) An orthonormal basis in  $\mathcal{F}$  is

$$E_1 = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Recall that for any triangular matrix, the eigenvalues are the numbers on the diagonal.

(g) If  $A$  is as described in the answer to (d), then it has only one eigenvalue,  $\alpha$ .

(h)  $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is an eigenvector for every  $A \in \mathcal{F}$ .

We remark that if  $\beta \neq 0$  then  $A = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \beta & \alpha \end{bmatrix}$  has no other independent eigenvectors.

(i) If  $A$  is as described in the answer to (d),

then  $A^* = \begin{bmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} \\ 0 & \bar{\alpha} & \bar{\beta} \\ 0 & 0 & \bar{\alpha} \end{bmatrix}$ . For  $A^* \in \mathcal{F}$  we must

have  $\beta = \gamma = 0$ , so that  $A = \alpha I$ .



## Question 2

(a)  $T$  is time-invariant. Its impulse response is  $g = (1, 2, 0, 0, 0, \dots)$  and its transfer function is  $G(z) = 1 + 2z^{-1} = \frac{z+2}{z}$ .

(b) The only pole of  $G$  is at  $z=0$ , and it is proper, hence it is stable. It is not strictly proper, since  $G(\infty) = 1$ . The system is FIR and its DC gain is  $G(1) = 3$ .

(c) Denote the right shift operator (delay by one step) by  $S$ . Then  $y = u + 2Su$ , hence (by the triangle inequality in  $\ell^p$ )  $\|y\|_p \leq \|u\|_p + 2\|Su\|_p = 3\|u\|_p$ .

(d) We have  $\hat{y}(z) = (1 + 2z^{-1})\hat{u}(z)$  and (by the Paley Wiener theorem)  $\|y\|_2 = \|\hat{y}\|_2$  (the last norm is in  $H^2(\mathcal{E})$ ). Thus  $\|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi} \int_{\mathcal{E}_1} |1 + 2z^{-1}|^2 \cdot |\hat{u}(z)|^2 \cdot |dz|$  where  $\mathcal{E}_1$  is the unit circle in  $\mathbb{C}$ .

Notice that  $1 + 2z^{-1}$  is on a circle centered at 1 and with radius 2, so that  $|1 + 2z^{-1}| \geq 1$  for  $z \in \mathcal{E}_1$ .

Hence  $\|y\|_2^2 \geq \frac{1}{2\pi} \int_{\mathcal{E}_1} |\hat{u}(z)|^2 \cdot |dz| = \|\hat{u}\|_2^2 = \|u\|_2^2$ .

(e) If  $y = Tu$  then  $\hat{y} = G\hat{u}$ . Since  $G(-2) = 0$ , we obtain  $\hat{y}(-2) = 0$ . Thus, we cannot obtain as  $Tu$  those signals  $w \in \ell^2$  for which  $\hat{w}(-2) \neq 0$ .

(f) On  $V = \{Tu \mid u \in \ell^2\}$  we define  $L$  by

$$u = Ly \iff \hat{u} = G^{-1}\hat{y} \iff y = Tu,$$

so that  $LT = I$ . According to the result of part (d),  $\|Ly\|_2 = \|u\|_2 \leq \|y\|_2$ , so that  $L$  is bounded from  $V$  to  $\ell^2$  and  $\|L\| \leq 1$  (as an operator in  $\mathcal{L}(V, \ell^2)$ ).

Now note that  $V$  is closed. Indeed, if  $(y_n)$  is a sequence in  $V$  and  $y_n \rightarrow y_0 \in \ell^2$ , then define  $u_n = Ly_n$ . It is clear that  $(u_n)$  is a Cauchy sequence in  $\ell^2$ , hence  $u_n \rightarrow u_0$  for some  $u_0 \in \ell^2$ . Since  $y_n = Tu_n$  and  $T$  is continuous, it follows that  $y_n \rightarrow Tu_0$ , so that  $y_0 = Tu_0 \in V$ . Thus,  $V$  contains its limit points.

We decompose  $\ell^2 = V + V^\perp$  and we define  $L$  on  $V^\perp$  in an arbitrary linear way, for example  $Lw = 0$  for all  $w \in V^\perp$ . Now  $L$  is defined on all of  $\ell^2$  and we still have that  $\|L\| \leq 1$ . (In fact, we have  $\|L\| = 1$ , but this is not important.)

If  $L$  were time-invariant (for some choice of its restriction to  $V^\perp$ ) then, according to the Fouré's - Segal theorem, we would have  $u = Ly$  iff  $\hat{u} = F\hat{y}$ , where  $F \in H^\infty(\mathbb{E})$ . Taking  $y \in V$  we see that we must have  $F = G^{-1}$ , but  $G^{-1}$  is unstable.

Question 3

(a)  $\langle f, g \rangle = \int_0^\infty f(t) \overline{g(t)} dt$ ,  
 $\|f\|_2 = \left( \int_0^\infty |f(t)|^2 dt \right)^{\frac{1}{2}}$ . If  $\varphi(t) = e^{-st}$ , where  $\operatorname{Re} s > 0$ , then  $\varphi \in L^2[0, \infty)$  and  $\|\varphi\|_2 = \frac{1}{\sqrt{2 \operatorname{Re} s}}$ .

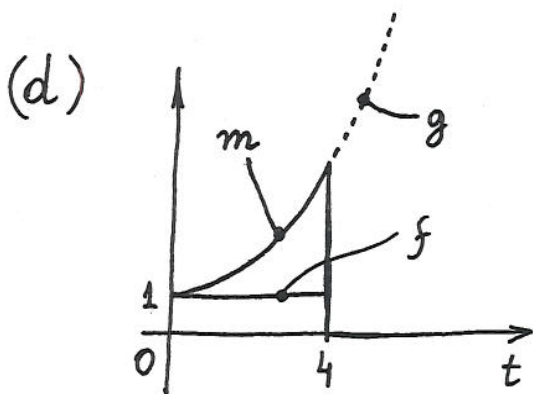
(b)  $u \in L^2[0, \infty)$ ,  $\hat{u}(s) = \int_0^\infty e^{-st} u(t) dt = \langle \varphi, u \rangle$ , where  $\varphi$  is as in part (a). By the Cauchy-Schwarz inequality,  $|\hat{u}(s)| \leq \|\varphi\|_2 \cdot \|u\|_2$ .

(c) If  $y \in L^1[0, \infty)$ , then

$$|\hat{y}(s)| = \left| \int_0^\infty e^{-st} y(t) dt \right| \leq \int_0^\infty |e^{-st}| \cdot |y(t)| dt.$$

If  $\operatorname{Re} s > 0$ , then  $|e^{-st}| < 1$  for all  $t > 0$ , so that

$$|\hat{y}(s)| \leq \int_0^\infty |y(t)| dt = \|y\|_1.$$



$$\hat{f}(s) = \frac{1}{s} (1 - e^{-4s}),$$

$$\hat{g}(s) = \frac{1}{s-5},$$

$$\hat{m}(s) = \hat{f}(s-5)$$

(e)  $\hat{g}$  is rational,

$\hat{f}$  and  $\hat{m}$  are not rational.

We have  $f, m \in L^1[0, \infty)$ ,

hence (by part (c))  $\hat{f}, \hat{m} \in H^\infty(\mathbb{C}_+)$ .  $\hat{f}$  and  $\hat{m}$

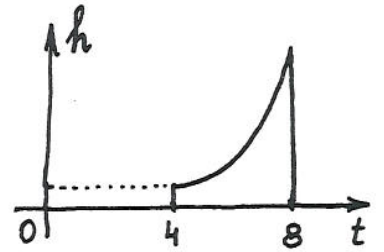
have no poles.  $\hat{g}$  has a pole at 5, hence it is not in  $H^\infty(\mathbb{C}_+)$ .

(f)  $h = S_4 m$ , hence  $\hat{h}(s) = e^{-4s} \hat{m}(s)$ , so that

$$\hat{h}(s) = \frac{e^{-4s}}{s-5} (1 - e^{20} e^{-4s}).$$

$$\|h\|_2 = \|m\|_2 = \left( \int_0^4 e^{10t} dt \right)^{\frac{1}{2}}$$

$$= \left( \frac{1}{10} e^{10t} \Big|_0^4 \right)^{\frac{1}{2}} = \frac{1}{\sqrt{10}} (e^{40} - 1)^{\frac{1}{2}}.$$



$\langle m, h \rangle = 0$  because  $m(t)h(t) = 0$  for almost every  $t \geq 0$ .

(g) According to the Paley-Wiener theorem (the continuous-time version), we have

$$\|\hat{m}\|_2 = \|m\|_2 = \frac{1}{\sqrt{10}} (e^{40} - 1)^{\frac{1}{2}},$$

$$\|\hat{h}\|_2 = \|h\|_2 = \frac{1}{\sqrt{10}} (e^{40} - 1)^{\frac{1}{2}},$$

$$\langle \hat{m}, \hat{h} \rangle = \langle m, h \rangle = 0.$$

### Question 4

(a) The system can be described by  
 $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t)$ , where

$$x(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & L \\ -L^* & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ H \end{bmatrix}$$

and  $C = [0 \ H^*]$ . Note that  $A^* = -A$ ,  $C = B^*$ .

(b)  $A^* = -A$  implies that  $iA$  is self-adjoint, so that  $iA$  has only real eigenvalues. Hence, all the eigenvalues of  $A$  are on  $i\mathbb{R}$ , so that  $A$  is not stable.

(c) We have  $\frac{d}{dt} \left( \frac{1}{2} \|p\|^2 \right) = \frac{1}{2} \frac{d}{dt} \langle p, p \rangle$

$$= \frac{1}{2} \left( \langle \dot{p}, p \rangle + \langle p, \dot{p} \rangle \right) = \frac{1}{2} \left( \langle \dot{p}, p \rangle + \overline{\langle \dot{p}, p \rangle} \right)$$

$$= \operatorname{Re} \langle \dot{p}, p \rangle, \quad \text{similarly for } q \text{ in place of } p,$$

hence

$$\dot{E} = \operatorname{Re} \left( \langle \dot{p}, p \rangle + \langle \dot{q}, q \rangle \right)$$

$$= \operatorname{Re} \left( \langle Lq, p \rangle - \langle L^* p, q \rangle + \langle Hu, q \rangle \right)$$

$$= \underbrace{\operatorname{Re} \left( \langle Lq, p \rangle - \overline{\langle Lq, p \rangle} \right)}_0 + \operatorname{Re} \langle u, H^* q \rangle$$

$$= \operatorname{Re} (u \bar{y}).$$

(d)  $G(s) = C(sI - A)^{-1}B$ . Applying the Laplace transformation to the system equations, with zero initial conditions, we get  $s\hat{p} = L\hat{q}$ ,  $s\hat{q} = -L^*\hat{p} + H\hat{u}$   
 hence  $s^2 \hat{q} = -L^* L \hat{q} + sH\hat{u}$ .

$$(s^2 I + L^* L) \hat{q} = s H \hat{u}, \quad \hat{y} = H^* \hat{q}, \quad \text{hence}$$

$$\boxed{G(s) = s H^* (s^2 I + L^* L)^{-1} H.}$$

(e) We have  $E(t) = \frac{1}{2} \|x(t)\|^2$ . If  $u=0$  then, according to the result from part (c),  $\dot{E}(t) = 0$ , so that  $\|x(t)\| = \|x(0)\|$ . Thus,  $e^{At}$  is isometric, hence  $\text{Ker } e^{At} = \{0\}$ , hence  $\det e^{At} \neq 0$ , hence  $e^{At}$  is invertible, hence  $e^{At}$  is unitary. (We remark that  $e^{At}$  is actually invertible for every square matrix  $A$ .)

$$\begin{aligned} (f) \quad G(s) + G(s)^* &= C(sI - A)^{-1} B + B^* (\bar{s}I - A^*)^{-1} C^* \\ &= C(sI - A)^{-1} B + C(\bar{s}I + A)^{-1} B \\ &= C \left[ (sI - A)^{-1} - (-\bar{s}I - A)^{-1} \right] B \\ &= (-\bar{s} - s) C(sI - A)^{-1} (-\bar{s}I - A)^{-1} B \\ &= (2 \operatorname{Re} s) C(sI - A)^{-1} (\bar{s}I - A^*)^{-1} B, \end{aligned}$$

this is the same as  $\overline{G(s)}$

so that we have  $k(s) = 2 \operatorname{Re} s$ .

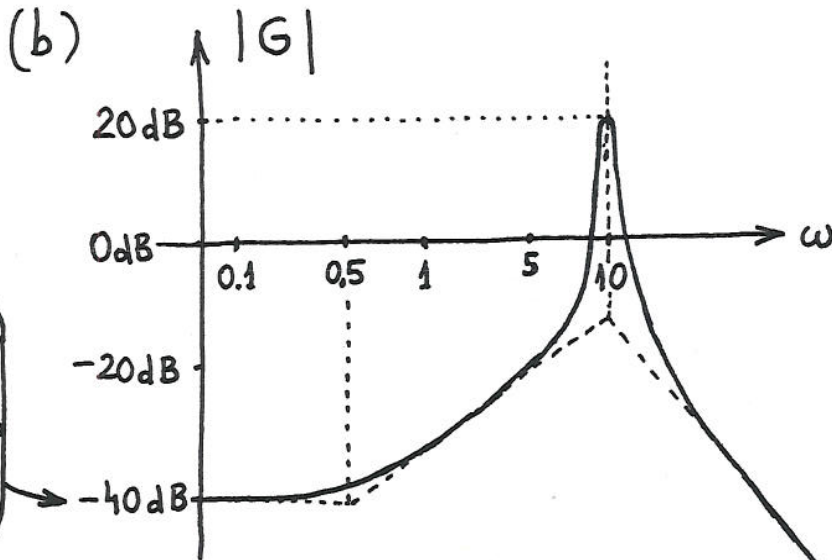
(g) Denote  $z(s) = (\bar{s}I - A^*)^{-1} B$ , so that  $z(s) \in \mathbb{C}^{4 \times 1}$ . We have, for  $\operatorname{Re} s > 0$ ,

$$\begin{aligned} \operatorname{Re} G(s) &= \frac{1}{2} [G(s) + G(s)^*] \\ &= (\operatorname{Re} s) z(s)^* z(s) \leftarrow \text{according to part (f)} \\ &= (\operatorname{Re} s) \|z(s)\|^2 > 0. \end{aligned}$$

In the last step we have used that  $H \neq 0 \Rightarrow B \neq 0 \Rightarrow z(s) \neq 0$ .

**Question 5** (2)  $G(s) = \frac{2s - 1}{s^2 + 0.2s + 100}$

$G$  is stable, because the denominator is a polynomial of degree two with positive coefficients.



Comments:  
 The zero at 0.5 causes the plot to rise at a slope of 20dB/dec. The pair of poles with absolute value  $\omega_n = \sqrt{100} = 10$  causes the linear approximation to the plot to bend down with -20dB/dec. The peak value is approximately at  $\omega_n = 10$ .

To estimate the peak value, we compute

$$G(10i) = \frac{20i - 1}{2i} = 10 + 0.5i,$$

so that  $|G(10i)| \approx 10$  (with an error less than 1%).

Thus,  $\|G\|_\infty \approx 10$ .

(c)  $u \in L^2(\mathbb{R})$  belongs to  $BL(\omega_b)$  if

$$(\mathcal{F}u)(i\omega) = 0 \text{ for } |\omega| > \omega_b.$$

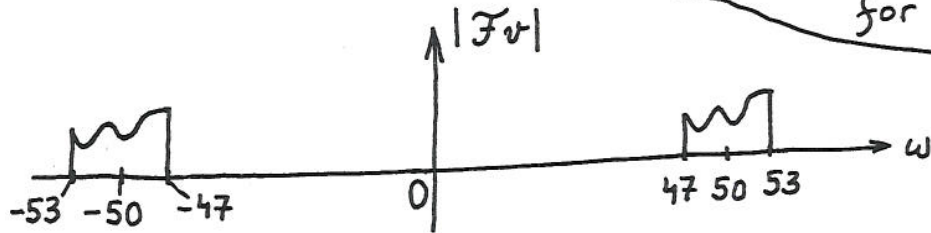
Here,  $\omega_b > 0$  and  $\mathcal{F}$  denotes the Fourier transform.

(d) 
$$e_k(t) = \frac{1}{\sqrt{\pi\omega_b}} \cdot \frac{\sin \omega_b(t - k\tau)}{\omega_b(t - k\tau)},$$

where  $k \in \mathbb{Z}$  and  $\tau = \frac{\pi}{\omega_b}$ . This basis is obtained as the inverse Fourier transform of the usual Fourier orthonormal basis in  $L^2[-i\omega_b, i\omega_b]$ .

(e) If  $u \in BL(3)$  and  $v(t) = u(t) \cos 50t$  then,

using that  $\cos 50t = \frac{1}{2} (e^{i50t} + e^{-i50t})$ ,  
 we obtain (as in amplitude modulation)  
 $(\mathcal{F}v)(i\omega) = \frac{1}{2} [(\mathcal{F}u)(i\omega - i50) + (\mathcal{F}u)(i\omega + i50)]$   
 so that  $(\mathcal{F}v)(i\omega) = 0$  for  $|\omega| > 53$  (and also  
 for  $|\omega| < 47$ ).



Thus,  $v \in BL(53)$ .

(f) The Fourier transformation from  $L^2(\mathbb{R})$  to  $L^2(i\mathbb{R})$  is isometric (this is the Parseval equality), hence  $\langle u, v \rangle = \langle \mathcal{F}u, \mathcal{F}v \rangle$ .  
 Since  $(\mathcal{F}v)(i\omega) = 0$  for  $|\omega| < 47$ , in particular for  $|\omega| < 3$ , we have  $\langle \mathcal{F}u, \mathcal{F}v \rangle = 0$   
 (because  $(\mathcal{F}u)(i\omega)(\mathcal{F}v)(i\omega)$  is zero for all  $\omega \in \mathbb{R}$ ).

(g) If  $u \in BL(3)$  is the input signal and  $y$  is the output signal, then  $(\mathcal{F}y)(i\omega) = G(i\omega)(\mathcal{F}u)(i\omega)$ .  
 Hence,  $(\mathcal{F}y)(i\omega) = 0$  for  $|\omega| > 3$ , so that  $y \in BL(3)$ .

Using again that  $\mathcal{F}$  is isometric, we have

$$\|y\|_2^2 = \|\mathcal{F}y\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |(\mathcal{F}y)(i\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-3}^3 |G(i\omega)|^2 \cdot |(\mathcal{F}u)(i\omega)|^2 d\omega. \quad \text{From the Bode}$$

plot in part (b) we see that  $|G(i\omega)| \leq 1$  for  $|\omega| < 3$ .

$$\text{Hence } \|y\|_2^2 \leq \frac{1}{2\pi} \int_{-3}^3 |(\mathcal{F}u)(i\omega)|^2 d\omega = \|\mathcal{F}u\|_2^2 = \|u\|_2^2.$$