E3.10
C1.6
ISE3.7

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

MSc and EEE/ISE PART III/IV: MEng, BEng and ACGI

**MATHEMATICS FOR SIGNALS AND SYSTEMS**

Friday, 7 May 10:00 am

Time allowed: 3:00 hours

**There are FIVE questions on this paper.**

**Answer THREE questions.**

*All questions carry equal marks*

Corrected Copy

**Any special instructions for invigilators and information for candidates are on page 1.**

Examiners responsible First Marker(s) : G. Weiss
Second Marker(s) : J.C. Allwright

**1.** Consider the matrix

$$T = \begin{bmatrix} 2 & 9 \\ 1 & 8 \\ 0 & 3 \end{bmatrix}.$$

(a) The range of $T$, denoted Ran $T$, is (by definition) the subspace of $\mathbb{C}^3$ consisting of all the vectors $Tx$, where $x \in \mathbb{C}^2$. Determine an orthonormal basis in Ran $T$. [4]

(b) Let $P$ be the projector from $\mathbb{C}^3$ onto Ran $T$. Determine an expression for the matrix corresponding to $P$, in terms of the basis computed in part (a). There is no need to do any numerical computations. [3]

(c) Recall that $T^*$ is the conjugate of the transpose of $T$. Compute the eigenvalues of $T^*T$. [3]

(d) Compute the norm of $T$, which is defined as $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$. [3]

(e) Define the matrix

$$P_0 = T(T^*T)^{-1}T^*.$$

Show that $P$ from part (b) is the same as $P_0$. Hint: first show that $P_0$ satisfies $P_0^2 = P_0$ and $P_0^* = P_0$, so that it is a projector. Then show that Ran $P_0$ = Ran $T$. [4]

(f) Give an example of a continuous-time stable matrix-valued transfer function $\mathbf{G}$ such that $\mathbf{G}(0) = T$ and $\mathbf{G}(\infty) = 0$. [3]

**2.** Consider the discrete-time signal $v = (v_k)$ defined by

$$v_k = \begin{cases} \frac{1}{k} & \text{if } k \text{ is even, } k \geq 2, \\ 0 & \text{else.} \end{cases}$$

We denote by $\hat{v}$ the $\mathcal{Z}$-transform of $v$.

(a) Indicate which of the following statements is true:

$$v \in L^2[0,\infty), \quad v \in l^1, \quad v \in l^2, \quad v \in l^\infty, \quad v \in \mathbf{c}_0. \qquad [3]$$

(b) Find a simple expression for $\frac{\mathrm{d}}{\mathrm{d}z}\hat{v}(z)$. [3]

(c) Where is $\hat{v}$ defined (i.e., the infinite sum which defines $\hat{v}$ is convergent)? Is $\hat{v}$ a rational function? Give a brief reasoning. [4]

(d) Find points on the unit circle where $\hat{v}(z)$ does not have a radial limit from the exterior. Hint: show that if $p$ is a pole of $\frac{\mathrm{d}}{\mathrm{d}z}\hat{v}(z)$ on the unit circle, then $v(z)$ tends to infinity as $z \to p$. [3]

(e) Consider the digital filter described by the difference equation

$$3y_k - y_{k-1} = u_k - 3u_{k-1},$$

where $u$ is the input and $y$ is the output. Is this filter time-invariant? Is this filter stable? Compute the transfer function $F$ of this filter. [3]

(f) Show that the input-output operator corresponding to the filter from part (e) is isometric. Hint: compute $|F(z)|$ for $|z| = 1$. [4]

**3.** For $1 \leq p < \infty$, we denote by $l^p$ the space of all sequences $u$ indexed by $k \in \{0, 1, 2, 3, ...\}$ for which $\Sigma_{k=0}^{\infty} |u_k|^p < \infty$. For such sequences $u$, we use the notation $||u||_p = \left(\Sigma_{k=0}^{\infty} |u_k|^p\right)^{\frac{1}{p}}$. We denote by $l^\infty$ the space of all bounded sequences, and let $||u||_\infty = \sup |u_k|$.

Suppose that $g \in l^1$ is the impulse response of a linear time-invariant discrete-time system $\Sigma$. The transfer function of $\Sigma$ is denoted by $\mathbf{G}$. Let $\mathbf{S}$ be the right shift operator on $l^p$, $1 \leq p \leq \infty$ (i.e., $\mathbf{S}$ acts as a delay by one step). Suppose that the initial state of $\Sigma$ is zero. Let $u \in l^p$ be the input signal to $\Sigma$ and let $y$ be the corresponding output signal.

(a) Compute $||\mathbf{S}||_{(p)} = \sup_{||u||_p \leq 1} ||\mathbf{S}u||_p$. [2]

(b) Show that

$$y = \sum_{k=0}^{\infty} g_k \mathbf{S}^k u. \qquad [4]$$

(c) Show that $y \in l^p$ and

$$||y||_p \leq ||g||_1 \cdot ||u||_p.$$

Hint: use the results from parts (a) and (b) and the triangle inequality in the space $l^p$, applied to an infinite sum. [4]

(d) Show that $\mathbf{G} \in H^\infty(\mathcal{E})$ ($\mathcal{E}$ denotes the exterior of the unit disc) and

$$||\mathbf{G}||_\infty \leq ||g||_1. \qquad [3]$$

(e) An averaging system $\mathbf{A}$ works as follows: if the input signal to $\mathbf{A}$ is $u \in l^2$, then the output signal is given by

$$y_k = \frac{1}{k+1} \sum_{j=0}^{k} u_j.$$

Show that $\mathbf{A}$ is bounded linear operator from $l^2$ to $l^3$, i.e., $||\mathbf{A}u||_3 \leq m||u||_2$ holds for some $m > 0$. Hint: Apply the Cauchy-Schwarz inequality to estimate $|y_k|$. [4]

(f) Is the operator $\mathbf{A}$ from part (e) time-invariant? Is $\mathbf{A}$ causal? (There is no need to explain your reasoning.) Hint: remember that causality means the following: $u_j$ has no influence on $y_k$ for $k < j$. [3]

4. Consider an electric circuit with two terminals which consists of resistors, capacitors and inductors. We denote by $I$ the current flowing into the circuit and by $U$ the voltage between the two terminals, with the directions taken such that $I(t)U(t)$ is the power absorbed at time $t$ ($t \geq 0$). We denote by $Z(s)$ the transfer function from $I$ to $U$ (also called the impedance of the circuit). The Laplace transforms of $I$ and $U$ are denoted by $\hat{I}$ and $\hat{U}$.

   (a) Give an example of a circuit for which $Z$ is not stable. Give a short argument why $Z$ cannot have poles in $\mathbb{C}_+$. [4]

   (b) In the sequel we assume that $Z$ is stable and has only simple poles (i.e., poles of order 1). Suppose that $I(t) = e^{i\omega t}$, where $\omega \in \mathbb{R}$ is fixed. (This current is not real, of course, but mathematically it is possible to consider it). Show that we have

$$U(t) = Z(i\omega)I(t) + e(t),$$

   where $\lim_{t\to\infty} e(t) = 0$. Hint: decompose $\hat{U}$ into partial fractions and show that only the fraction that has a pole at $i\omega$ matters, all the others add up to $\hat{e}$. [5]

   (c) Show that if $I(t) = \cos(\omega t)$, then

$$U(t) = U_0 \cos(\omega t + \varphi) + e_r(t),$$

   where $U_0 = |Z(i\omega)|$, $\varphi = \arg Z(i\omega)$ and $\lim_{t\to\infty} e_r(t) = 0$. Hint: take real parts in the result of part (b). [4]

   (d) For $I$ as in part (c) and $e_r(t) = 0$ (this is true for large $t$) compute the average power absorbed by the circuit. Hint: take the average over one period of $I$. [3]

   (e) Show that $\varphi$ from part (c) must be in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Explain why Re $Z(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Hint: use the result from part (d). [4]

5. Consider the following functions defined for $s \in \mathbb{C}_+$:

$$g_1(s) = \frac{s^2 - s + 100}{s^2 + s + 100}, \quad g_2(s) = \frac{1}{s+2}, \quad g_3(s) = \frac{e^{-0.4s}}{s^2 + 60}, \quad g_4(s) = e^{-7s},$$

$$g_5(s) = \frac{\operatorname{Re} s}{s+1}, \quad g_6(s) = \frac{s-3}{s+3}, \quad g_7(s) = \frac{3s}{s^2 - 11}, \quad g_8(s) = s^{-\frac{1}{3}},$$

$$g_9(s) = \frac{2s - 4}{s^2 + s + 100}.$$

(a) Which of the functions listed above is in $H^2(\mathbb{C}_+)$? [3]

(b) Which of the functions listed above is in $H^\infty(\mathbb{C}_+)$? For those that you find, estimate their norm in $H^\infty(\mathbb{C}_+)$, with a precision of $\pm 5\%$. [3]

(c) Which of the functions you found in part (b), when regarded as a transfer function, determines an isometric operator from input signals in $L^2[0, \infty)$ to output signals in $L^2[0, \infty)$? [3]

(d) Compute the inverse Laplace transforms of $g_2, g_3$ and $g_6$. [3]

(e) Show that for every $\varphi \in H^2(\mathbb{C}_+)$ the following formula holds:

$$\langle \varphi, g_2 \rangle = \varphi(2).$$

Here, the inner product is taken in $H^2(\mathbb{C})$. Hint: Use the inverse Laplace transforms of $\varphi$ and $g_2$ and the Paley-Wiener theorem. [3]

(f) Assume that $u \in L^2_{loc}[0, \infty)$ is periodic with period $\tau > 0$. State the Fourier series expansion of $u$ in terms of complex exponentials and give the formula for the computation of the Fourier coefficients. [3]

(g) Compute the Laplace transform $\hat{u}$ for $u$ as in part (f), in terms of the Fourier coefficients of $u$. Compute the poles of $\hat{u}$. [2]

[ END ]

Maths - 7.4.04 ~~13/4/04~~ E3.10 ISE 3.7 C 1.6

# Mathematics for Signals and Systems

## Exam of May 2004

## SOLUTIONS

**Question 1** (a) We normalize the first column of T: $e_1 = \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, where $\alpha = \frac{1}{\sqrt{5}}$, so that $\|e_1\| = 1$. Now we take $e_2 = \beta \left( \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right)$. To make sure that $\langle e_1, e_2 \rangle = 0$, we have to take

$$\gamma = -\left\langle \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\rangle \Big/ \left\langle \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\rangle = -\frac{26}{5},$$

so that $e_2 = \beta \begin{bmatrix} 9 - 52/5 \\ 8 - 26/5 \\ 3 \end{bmatrix} = \beta \begin{bmatrix} -1.4 \\ 2.8 \\ 3 \end{bmatrix}$. For $\|e_2\| = 1$, we take $\beta = 1 \Big/ \sqrt{(1.4)^2 + (2.8)^2 + 3^2}$, $\beta = 0.230632\ldots$, so that finally

$$e_2 = \begin{bmatrix} -0.32288\ldots \\ 0.64577\ldots \\ 0.69189\ldots \end{bmatrix}$$

Now $\{e_1, e_2\}$ is an orthonormal basis in Ran T.

(b) $Px = e_1 \langle x, e_1 \rangle + e_2 \langle x, e_2 \rangle$

$$= (e_1 e_1^* + e_2 e_2^*)\, x,$$

so that $\qquad P = e_1 e_1^* + e_2 e_2^*.$

A numerical computation (which is not required) gives

$$P = \begin{bmatrix} 0.904255\ldots & 0.19149\ldots & -0.223404\ldots \\ 0.19149\ldots & 0.617021\ldots & 0.446808\ldots \\ -0.223404\ldots & 0.446808\ldots & 0.478723\ldots \end{bmatrix}.$$

(c) $$T^*T = \begin{bmatrix} 5 & 26 \\ 26 & 154 \end{bmatrix},$$

$$\det(\lambda I - T^*T) = \lambda^2 - 159\lambda + 94,$$

so that the eigenvalues of $T^*T$ are $\lambda_{1,2} = \dfrac{159 \pm \sqrt{24{,}905}}{2}$

$$\lambda_1 = 0.593409\ldots, \quad \lambda_2 = 158.4065\ldots$$

(d) $\|T\|^2 = \max\{\lambda_1, \lambda_2\} = 158.4065\ldots,$

so that $\quad \|T\| = 12.5859\ldots$

(e) $P_0^2 = T(T^*T)^{-1}\underbrace{T^*T(T^*T)^{-1}}_{I}T^* = P_0$.

$P_0^* = \overbrace{T^{**}}^{T}(T^*T)^{-1*}T$. Since $T^*T$ is self-adjoint, its inverse is also self-adjoint, and we obtain $P_0^* = P_0$. Hence, $P_0$ is a projector. If $z = P_0 x$ for some $x \in \mathbb{C}^3$, i.e., $z \in \operatorname{Ran} P_0$, then

$z = T\underbrace{(T^*T)^{-1}T^*x}_{y} = Ty$, $y \in \mathbb{C}^2$,

so that $z \in \operatorname{Ran} T$. Every $y \in \mathbb{C}^2$ can be obtained as above, because $\operatorname{Ran} T^* = \mathbb{C}^2$ and $(T^*T)^{-1}$ is invertible. Hence, $\operatorname{Ran} P_0 = \operatorname{Ran} T$.

(f) One possible example is

$$G(s) = \frac{1}{s+1}T.$$

$\boxed{\text{Question 2}}$ (a) $v \in l^2 \Rightarrow v \in c_0$
$\Rightarrow v \in l^\infty.$

(b) $\hat{v}(z) = \sum_{n=1}^{\infty} \frac{1}{2n} z^{-2n}$, hence

$$\frac{d}{dz}\hat{v}(z) = -\sum_{n=1}^{\infty} z^{-2n-1} = \frac{-1}{z^3}\sum_{l=0}^{\infty}\left(\frac{1}{z^2}\right)^l$$

$$= \frac{-1}{z^3}\cdot\frac{1}{1-\frac{1}{z^2}} = \frac{1}{z(1-z^2)}.$$

(c) The sum defining $\hat{v}$ is a power series in the variable $z^{-2}$, hence the largest open domain where it converges consists of those $z \in \mathbb{C}$ for which $|z^{-2}| < R$, where $\frac{1}{R} = \limsup \left(\frac{1}{2n}\right)^{\frac{1}{n}}$. We get $\frac{1}{R} = \lim e^{-\frac{1}{n}\ln(2n)} = e^0 = 1$, $R = 1$.

Thus, the power series converges if $|z^{-2}| < 1$, equivalently, $|z| > 1$, i.e., $z \in \mathcal{E}$.

A second argument which yields the same result is the following: $v \in l^2$, hence by the Paley-Wiener theorem $\hat{v} \in H^2(\mathcal{E})$, in particular, $\hat{v}$ is defined on $\mathcal{E}$. This argument does not tell us if the series converges on any larger domain (but we know from the first argument that this is not the case). We can decompose $\frac{d}{dz}\hat{v}$ into partial fractions:

$$\frac{d}{dz}\hat{v}(z) = \frac{1}{z(1-z^2)} = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{z+1}$$

(we have $a=1$, $b=c=-\frac{1}{2}$, but these values are not important). Hence

$$\hat{v}(z) = \ln\left[ z^a (z-1)^b (z+1)^c \right] + K,$$

so that $\hat{v}$ is not rational. (We remark that $K=0$ and $\hat{v}$ has an analytic continuation into $\mathcal{D}$ except for the real segment $[-1,1]$.)

(d) For $z=\pm 1$, $\hat{v}$ has no radial limit at $z$, since it tends to infinity according to the formula for $\hat{v}$ derived above.

(e) The filter is time-invariant and stable, and its transfer function is

$$F(z) = \frac{1-3z^{-1}}{3-z^{-1}} = \frac{1}{3}\cdot\frac{z-3}{z-\frac{1}{3}}.$$

(f) Take $z$ on the unit circle,

$$z = \cos\varphi + i\sin\varphi, \quad \varphi \in \mathbb{R}.$$

Then

$$|F(z)|^2 = \frac{1}{9} \cdot \frac{|z-3|^2}{\left|z-\frac{1}{3}\right|^2}$$

$$= \frac{1}{9} \cdot \frac{(\cos\varphi - 3)^2 + (\sin\varphi)^2}{\left(\cos\varphi - \frac{1}{3}\right)^2 + (\sin\varphi)^2}$$

$$= \frac{1}{9} \cdot \frac{(\cos\varphi)^2 - 6\cos\varphi + 9 + (\sin\varphi)^2}{(\cos\varphi)^2 - \frac{2}{3}\cos\varphi + \frac{1}{9} + (\sin\varphi)^2}$$

$$= \frac{1}{9} \cdot \frac{10 - 6\cos\varphi}{\frac{10}{9} - \frac{6}{9}\cos\varphi} = \frac{1}{9} \cdot 9 = 1.$$

Let $T$ be the input-output operator corresponding to $F$. Then for any $u \in \ell^2$,

$$\| T u \|_2 = \| F \cdot \hat{u} \|_2$$

$$= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|F(e^{i\varphi})|^2}_{1} \cdot |\hat{u}(e^{i\varphi})|^2 \, d\varphi \right)^{\frac{1}{2}}$$

$$= \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{u}(e^{i\varphi})|^2 \, d\varphi \right)^{\frac{1}{2}} = \| \hat{u} \|_2 = \| u \|_2$$

(we have used twice the Paley-Wiener theorem).

## Question 3

(a) We have $\|Su\|_p = \|u\|_p$ for all $1 \leq p \leq \infty$, so that $\|S\|_{(p)} = 1$ (this is the operator norm from $l^p$ to itself).

(b) We have
$$y_n = (g * u)_n = \sum_{k=0}^{n} g_k u_{n-k}$$
$$= \sum_{k=0}^{\infty} g_k (S^k u)_n ,$$
so that the stated formula holds.

(c) We have
$$\|y\|_p = \left\| \sum_{k=0}^{\infty} g_k S^k u \right\|_p$$

by the triangle inequality
$$\leq \sum_{k=0}^{\infty} \|g_k S^k u\|_p$$
$$\leq \sum_{k=0}^{\infty} |g_k| \cdot \|S^k\|_{(p)} \cdot \|u\|_p$$

It is easy to see that $\|S^k\|_{(p)} = 1$ (by the same argument as in part (a)), so that
$$\|y\|_p \leq \left( \sum_{k=0}^{\infty} |g_k| \right) \cdot \|u\|_p .$$

This is exactly the required estimate.

(d) For $z \in \mathcal{E}$ (i.e., for $|z^{-1}| < 1$) we have
$$|G(z)| = \left| \sum_{k=0}^{\infty} g_k z^{-k} \right| \leq \sum_{k=0}^{\infty} |g_k| \cdot \underbrace{|z^{-k}|}_{<1} \leq \|g\|_1 .$$

It follows that

$$\|G\|_\infty = \sup_{z \in \mathcal{E}} |G(z)| \leq \|g\|_1 .$$

(e) We have
$$|y_k| = \left| \sum_{j=0}^{k} \frac{1}{k+1} \cdot u_j \right|$$

$$\leq \left( \sum_{j=0}^{k} \frac{1}{(k+1)^2} \right)^{\frac{1}{2}} \cdot \left( \sum_{j=0}^{k} |u_j|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( \frac{k+1}{(k+1)^2} \right)^{\frac{1}{2}} \cdot \left( \sum_{j=0}^{\infty} |u_j|^2 \right)^{\frac{1}{2}}, \quad \text{so that}$$

$$|y_k| \leq \frac{1}{\sqrt{k+1}} \cdot \|u\|_2 .$$

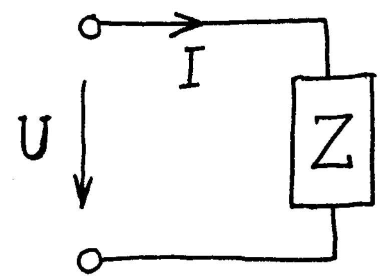
Since the sequence $\left( \frac{1}{\sqrt{k+1}} \right)$ is in $\ell^3$, it follows that $y \in \ell^3$ and

$$\|y\|_3 \leq \underbrace{\left\| \left( \frac{1}{\sqrt{k+1}} \right) \right\|_3}_{m} \cdot \|u\|_2 .$$

We remark that in fact, $\mathbf{A}$ is a bounded operator also from $\ell^2$ to $\ell^2$, but this is more difficult to prove (see Rudin, p. 75).

(f) It is clear that $\mathbf{A}$ is causal. It is not time-invariant – to see this, compare $\mathbf{A}u$ and $\mathbf{A}Su$, where $u = (1, 0, 0, 0, \ldots)$.

## Question 4




(a) If the circuit is a capacitor, then $Z(s) = \frac{1}{Cs}$, which is proper but not stable. If the circuit is an inductor, then $Z(s) = Ls$, which is not proper, hence not a stable transfer function. If $Z$ would have poles in $\mathbb{C}_+$, then the voltage produced by an initial state (with $I = 0$) would grow without bound. This is impossible, since there are no sources in the circuit.

A more rigorous (but longer) argument is that if there would be a pole in $\mathbb{C}_+$, then (by continuity) there would still be a pole in $\mathbb{C}_+$ if we connected a large resistor in parallel to the circuit. The voltage on this resistor (with no current from the exterior) would grow without bound, hence the circuit would be an infinite energy source (dissipated on the resistor), which is physically impossible.

(b) $Z$ is stable, has simple poles, $I(t) = e^{i\omega t} \implies \hat{I}(s) = \frac{1}{s - i\omega}$. Decompose $Z$ into simple fractions: $Z(s) = \sum_{k=1}^{N} \frac{r_k}{s - p_k}$ where $p_k \in \mathbb{C}_-$, $r_k \in \mathbb{C}$. The voltage due to the initial state has the form

$$\hat{U}_0(s) = \sum_{k=1}^{N} \frac{u_k}{s - p_k} \implies U_0(t) = \sum_{k=1}^{N} u_k e^{p_k t},$$

so that $\lim_{t \to \infty} U_0(t) = 0$. We have

$$\hat{U}(s) = Z(s)\,\hat{I}(s) + \hat{U}_0(s)$$

$$= \left( \sum_{k=1}^{N} \frac{r_k}{s - p_k} \right) \cdot \frac{1}{s - i\omega} + \hat{U}_0(s).$$

Decomposing into simple fractions,

$$\hat{U}(s) = \frac{Z(i\omega)}{s - i\omega} + \underbrace{\sum_{k=1}^{N} \frac{q_k}{s - p_k} + \hat{U}_0(s)}_{\hat{e}(s)},$$

where the coefficients $q_1, \dots q_N$ are not worth computing. Taking the inverse Laplace transformation,

$$U(t) = Z(i\omega)\,e^{i\omega t} + e(t).$$

All the poles of $\hat{e}$ are in $\mathbb{C}_-$, hence $e(t) \to 0$ as $t \to \infty$.

(c) Taking real parts in the result proved in (b), and denoting $e_r(t) = \mathrm{Re}\, e(t)$, we obtain that if $I(t) = \cos \omega t$, then

$$U(t) = \mathrm{Re}\left(Z(i\omega)\, e^{i\omega t}\right) + e_r(t).$$

Denoting $U_0 = |Z(i\omega)|$, $\varphi = \arg Z(i\omega)$, we have $Z(i\omega)\, e^{i\omega t} = U_0\, e^{i(\omega t + \varphi)}$, so that $\mathrm{Re}\left(Z(i\omega) e^{i\omega t}\right) = U_0 \cos(\omega t + \varphi)$.

(d) The average power over one period is
$P_{ave} = \frac{1}{\tau} \int_0^{\tau} \cos(\omega t)\, U_0 \cos(\omega t + \varphi)\, dt$. Since

$\tau = \frac{2\pi}{\omega}$,

$$P_{ave} = \frac{\omega U_0}{2\pi} \int_0^{2\pi/\omega} \cos^2(\omega t) \cos\varphi \, dt - \frac{\omega U_0}{2\pi} \int_0^{2\pi/\omega} \cos(\omega t) \sin(\omega t) \sin\varphi \, dt.$$

The second integral is zero, and we obtain

$$P_{ave} = \frac{1}{2} U_0 \cos\varphi.$$

(e) Since the circuit contains no energy sources, we must have $P_{ave} \geqslant 0$, hence $\cos\varphi \geqslant 0$, hence $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Since $\mathrm{Re}\, Z(i\omega) = U_0 \cos\varphi$, it follows that $\mathrm{Re}\, Z(i\omega) \geqslant 0$. This is true for all $\omega \in \mathbb{R}$.

**Question 5** (a) $g_2$ and $g_9$ are in $H^2$.
(For $g_8$, the problem is that the integral $\int_{-\infty}^{\infty} |g_8(i\omega)|^2 d\omega$ diverges for large $|\omega|$.)

(b) $g_1, g_2, g_4, g_6$ and $g_9$ are in $H^\infty(\mathbb{C}_+)$.

$\|g_1\|_\infty = 1, \quad \|g_2\|_\infty = |g_2(0)| = \frac{1}{2}, \quad \|g_4\|_\infty = 1,$

$\|g_6\|_\infty = 1, \quad \|g_9\|_\infty \approx |g(10i)| \approx \sqrt{4.16} \approx 2.039$

(to see that the peak of $g_9$ is around $s = 10i$, sketch the Bode magnitude plot of $g_9$).

(c) For a transfer function to determine an isometric input-output operator, it must have the property

$$|g(i\omega)| = 1 \quad \text{for all} \quad \omega \in \mathbb{R}$$

(see also part (f) of Question 2).
This property is shared by $g_1, g_4, g_6$.

(d) $(\mathcal{L}^{-1} g_2)(t) = e^{-2t} \quad \mathcal{L}(\sin\sqrt{60}\,t)(s) = \dfrac{\sqrt{60}}{s^2 + 60},$

hence $(\mathcal{L}^{-1} g_3)(t) = \dfrac{1}{\sqrt{60}} \sin\left[\sqrt{60}\,(t - 0.4)\right],$

$g_6(s) = 1 - \dfrac{6}{s+3}$, hence $(\mathcal{L}^{-1} g_6)(t) = \delta_0(t) - 6e^{-3t}$.

(e) We denote $f = \mathcal{L}^{-1}\varphi$, $h = \mathcal{L}^{-1} g_2$, so that $h(t) = e^{-2t}$. Then

$$\langle \varphi, g_2 \rangle = \langle f, h \rangle = \int_0^\infty f(t)\, e^{-2t}\, dt$$
$$= \hat{f}(2) = \varphi(2).$$

(f) If $u \in L^2_{loc}[0,\infty)$ is periodic with period $\tau$, then on each period it is an $L^2$ function, hence it can be written as a Fourier series. We use on $L^2[0,\tau]$ the orthonormal basis

$$\left\{ e_k \;\middle|\; e_k(t) = \frac{1}{\sqrt{\tau}}\, e^{ik\frac{2\pi}{\tau}t},\quad k \in \mathbb{Z} \right\}$$

(note that $\frac{2\pi}{\tau}$ is the fundamental frequency in rad/sec). The Fourier expansion of $u$ is $u = \sum_{k\in\mathbb{Z}} c_k e_k$, where $c_k = \langle u, e_k \rangle$

$$= \frac{1}{\sqrt{\tau}} \int_0^\tau u(t)\, e^{-ik\frac{2\pi}{\tau}t}\, dt .$$

(g) Applying the Laplace transformation to the Fourier series of $u$, we get

$$\hat{u}(s) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{\tau}} \cdot \frac{1}{s - ik\frac{2\pi}{\tau}} .$$

Thus, the poles of $\hat{u}$ are those numbers $ik\, 2\pi/\tau$ ($k \in \mathbb{Z}$) for which $c_k \neq 0$.