IMPERIAL COLLEGE LONDON

E3.09
C1.0
ISE3.9

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS ~~2007~~

EEE/ISE PART III/IV: MEng, BEng and ACGI

Sample exam paper – December 2006

CONTROL ENGINEERING

Friday, ~~27 April 2:30~~ pm

Time allowed: 3:00 hours

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	A. Astolfi
	Second Marker(s) :	C. Ling

EJ.09 | ISE 3.9

December 2006

Control engineering – Sample exam paper

1. Consider the linear, discrete-time system described by

$$x(k+1) = Ax(k) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(k).$$

a) Let

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Compute $x(k)$ for $k = 1,2,3,4$. Sketch the trajectory on the state space.[2 marks]

b) Let

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Compute $x(k)$ for $k = 1,2,3,4$. Sketch the trajectory on the state space.[2 marks]

c) Let

$$x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Exploiting the results in parts a) and b), compute the trajectory of the system for $k = 1,2,3,4$. [4 marks]

d) Exploting the results in parts a), b) and c) and the definition of stability show that the zero equilibrium of the system is stable, but not attractive. Is the stability property uniform? [8 marks]

e) The considered discrete-time system is the Euler approximate model, with sampling time $T = 1$, of a continuous-time system described by the equation

$$\dot{x} = A_c x.$$

Determine the matrix A_c and discuss the stability properties of this continuous-time system. [4 marks]

2. Consider the nonlinear model of an AC/DC converter, given by

$$\dot{x}_1 = -\frac{1}{L}x_2 u + \frac{E}{L}, \qquad \dot{x}_2 = -\frac{1}{RC}x_2 + \frac{1}{C}x_1 u,$$

with $x_1(t) \in I\!R$, $x_2(t) \in I\!R$, $u(t) \in I\!R$, E, R, L and C positive constants.

a) Let $u = u_0 > 0$, with u_0 constant, and compute the equilibrium point of the system. [4 marks]

b) Write the linearized model of the system around the equilibrium point computed in part a). [6 marks]

c) Study the stability properties of the linearized model determined in part b). [4 marks]

d) Using the principle of stability in the first approximation discuss the stability properties of the equilibrium of the nonlinear system. [2 marks]

e) To *boost* the energy of the system it is possible to select $u(t) = 0$, for all t. Study the behaviour of the system in this situation. In particular show that the system does not have any equilibrium and that $\lim_{t\to\infty} x_1(t) = +\infty$ and that $\lim_{t\to\infty} x_2(t) = 0$. [4 marks]

3. Consider the linear, discrete-time system described by

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} u(k).$$

a) Compute the reachability matrix of the system, discuss its reachability properties, and determine a basis for its reachable space. [4 marks]

b) Compute the set of reachable states in one step and in two steps. [4 marks]

c) Show that the system is controllable in two steps. [6 marks]

d) Write the system in the canonical form for non-reachable systems. In particular, show that the unreachable subsystem is described by the equation

$$\hat{x}_2(k+1) = 0.$$

[6 marks]

4. Consider the linear, continuous-time system

$$\dot{x} = Ax + Bu \quad = \quad \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u,$$

with $x(t) \in \mathbb{R}^3$ and $u(t) \in \mathbb{R}$. Consider a reference signal $w(t) = [w_1(t), w_2(t), w_3(t)]'$ and consider the problem of designing a feedback control law such that the state of the closed-loop system asymptotically tracks the signal w.

a) The class of reference signals $w(t)$ for which asymptotic tracking is achievable is characterized by the condition

$$\dot{w} = Aw.$$

Show that $w(t) = [\sin t, \cos t, 0]'$ belongs to this class. [5 marks]

b) Suppose $w(t)$ is such that the asymptotic tracking problem is solvable. To design a control law which solves the asymptotic tracking problem one could proceed as follows.

i) Define the tracking error $e = x - w$ and write a differential equation for e. In particular, show that e is such that

$$\dot{e} = Ae + Bu.$$

[5 marks]

ii) Design a control law $u = Ke$, with K such that the matrix $A + BK$ has all eigenvalues with negative real part. In particular, select K such that all eigenvalues of $A + BK$ are equal to -1. [6 marks]

iii) Show that there are infinitely many matrices K assigning the eigenvalues of the closed-loop system as required and discuss why this is the case. [4 marks]

5. A linear, discrete-time system is described by the equations

$$\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) + u(k) \\ x_2(k+1) &= \alpha x_1(k) + x_2(k) + u(k) \\ y(k) &= x_1(k) - x_2(k) \end{aligned}$$

where $x(k) = [x_1(k), x_2(k)]' \in \mathbb{R}^2$, $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}$ and α is a constant parameter.

a) Study the reachability, controllability and stabilizability properties of the system as a function of α. [4 marks]

b) Study the observability and detectability properties of the system as a function of α. [4 marks]

c) Assume $\alpha \neq 1$. Design a dead-beat output feedback controller applying the separation principle. In particular, select the state feedback gain K such that the matrix $(A+BK)$ has two eigenvalues equal to 0 and the output injection gain L such that the matrix $(A+LC)$ has two eigenvalues equal to 0. Note that K and L may depend on α. [8 marks]

d) Compute

$$\lim_{\alpha \to 1} \|K\| \qquad \lim_{\alpha \to 1} \|L\|$$

and explain your results using the solutions of parts a) and b). [4 marks]

6. Consider the linear electric network in Figure 6.1(a). Let $R > 0$, $C > 0$ and $L > 0$. Denote by u the driving voltage, by x_1 the voltage across the capacitor C, by x_2 the current through the inductor L, and by y the current through the voltage source.

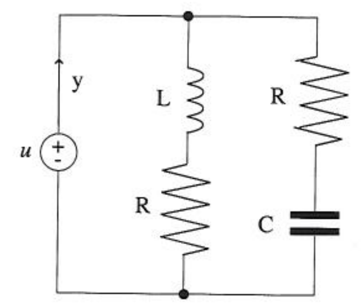

Figure 6.1(a)

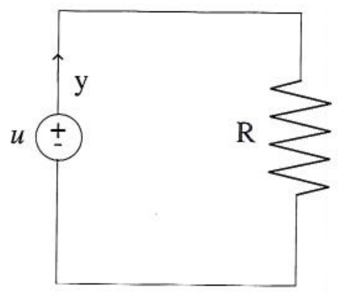

Figure 6.1(b)

a) Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in state-space form, regarding u as the input and y as the output. [4 marks]

b) Study the reachability and stabilizability properties of the dynamical system determined in part a). [4 marks]

c) Study the observability and detectability properties of the dynamical system determined in part a). [4 marks]

d) Show that if $R^2C = L$ then the unreachable subsystem is observable, and the unobservable subsystem is reachable. Hence conclude that the system does not have a reachable and observable subsystem. [4 marks]

e) Show that if $R^2C = L$ then the input-output behaviours of the circuits in Figures 6.1(a) and 6.1(b) are the same. [4 marks]

Dec. 06

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Control engineering sample exam paper - Model answers

Question 1

a) By a direct computation we obtain

$$x(1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad x(2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad x(3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x(4) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x(0).$$

This trajectory is sketched in Figure 1 (left). Note that $A^2 = -I$ and $A^4 = I$.

b) By a direct computation we obtain

$$x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x(2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad x(3) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad x(4) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x(0).$$

This trajectory is sketched in Figure 1 (right).

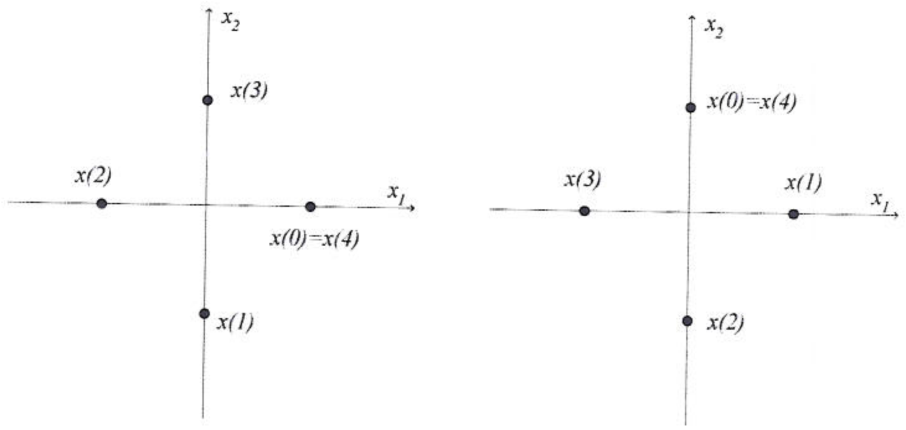

Figure 1: Sketch of the trajectories of the considered system.

c) Using the results in a) and b) yields

$$x(1) = \begin{bmatrix} \beta \\ -\alpha \end{bmatrix}, \quad x(2) = \begin{bmatrix} -\alpha \\ -\beta \end{bmatrix}, \quad x(3) = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}, \quad x(4) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = x(0).$$

This trajectory is sketched in Figure 2.

d) We have to show that for any $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that $\|x(0)\| < \delta(\epsilon)$ implies $\|x(k)\| < \epsilon$ for all k. Note that for any initial condition the distance of $x(k)$ from the origin is constant, i.e. $\|x(k)\|$ is constant for all k. Therefore, the selection $\delta(\epsilon) = \epsilon$ makes the above implication true. The state does not converge to the origin, hence the equilibrium is not attractive. Finally, the system is time-invariant, hence stability is uniform.

e) Given a continuous-time system $\dot{x} = A_c x$, its Euler approximate model, with sampling time T, is

$$x(k+1) = (I + TA_c)x(k) = Ax(k).$$

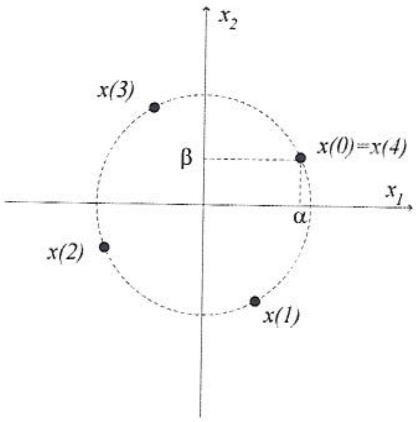

Figure 2: Sketch of a generic trajectory of the considered system.

Hence,

$$A_c = \frac{A - I}{T} = \frac{1}{T}\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The characteristic polynomial of the matrix A_c is $s^2 + 2s + 2$, hence the continuous-time system is asymptotically stable. (Note that, in this case, this is true regardless of the value of T.)

Question 2

a) The equilibrium points are the solutions of the equations

$$0 = -\frac{1}{L}x_2 u_0 + \frac{E}{L} \qquad\qquad 0 = -\frac{1}{RC}x_2 + \frac{1}{C}x_1 u_0.$$

From the first equation we have

$$x_2 = \frac{E}{u_0}$$

and from the second equation

$$x_1 = \frac{1}{Ru_0}x_2 = \frac{E}{Ru_0^2}.$$

b) The linearized model is given by

$$\dot{\delta}_x = A\delta_x + B\delta_u = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} \delta_x + \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial u} \\ \frac{\partial \dot{x}_2}{\partial u} \end{bmatrix} \delta_u = \begin{bmatrix} 0 & -\frac{u_0}{L} \\ \frac{u_0}{C} & -\frac{1}{RC} \end{bmatrix} \delta_x + \begin{bmatrix} -\frac{E}{Lu_0} \\ \frac{E}{CRu_0^2} \end{bmatrix} \delta_u.$$

c) The characteristic polynomial of the matrix A is

$$p(s) = s^2 + \frac{1}{RC}s + \frac{u_0^2}{LC},$$

hence, by Routh test, its roots have negative real part. This implies that the linearized system is asymptotically stable for any positive R, L, C and u_0.

d) By the principle of stability in the first approximation, the equilibrium of the nonlinear system is locally asymptotically stable.

e) If $u = 0$ we have

$$\dot{x}_1 = \frac{E}{L}, \qquad\qquad \dot{x}_2 = -\frac{1}{RC}x_2.$$

The system does not have any equilibrium because the equation $0 = E$ does not have any solution. Moreover, $x_1(t) = x_1(0) + Et$ and $x_2(t) = e^{-\frac{t}{RC}}x_2(0)$ which shows that $x_1(t) \to \infty$ and $x_2(t) \to 0$.

Question 3

a) The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 2 & -2 & -2 \end{bmatrix}.$$

The matrix R has rank 2. In fact, its determinant is zero, and the first two columns are linearly independent. This implies that the system is not reachable. A basis for the reachable space is given by the image of R, namely

$$\text{Im}R = \text{Im} \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 2 & -2 \end{bmatrix}.$$

b) The set of reachable states in one step is given by

$$x(1) = Bu(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} u(0),$$

with $u(0) \in I\!R$. The set of reachable states in two steps is given by

$$x(2) = ABu(0) + Bu(1) = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix},$$

with $u(0) \in I\!R$ and $u(1) \in I\!R$.

c) The condition for controllability in two steps is $\text{Im}A^2 \subseteq \text{Im}[B, AB]$, which is equivalent to

$$\text{rank} \begin{bmatrix} B & AB \mid A^2 \end{bmatrix} = \text{rank} \begin{bmatrix} B & AB \end{bmatrix} = 2.$$

Note that

$$\begin{bmatrix} B & AB \mid A^2 \end{bmatrix} = \left[\begin{array}{cc|ccc} 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 2 & -2 & -2 & 0 & 0 \end{array}\right]$$

and this has rank two. Hence the system is controllable in two steps.

d) To write the system is the canonical form for unreachable systems we define a matrix L from the first two columns of R and adding a third colum which makes it invertible, namely

$$L = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}.$$

The transformed system is described by

$$\hat{x}(k+1) = L^{-1}AL\hat{x}(k) + L^{-1}Bu(k) = \left[\begin{array}{cc|c} 0 & -1 & 1/2 \\ 1 & 0 & 1/2 \\ \hline 0 & 0 & 0 \end{array}\right] \hat{x}(k) + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u(k),$$

from which we see that the unreachable system is described by the equation $\hat{x}_2(k+1) = 0$. This is consistent with the fact that the unreachable system has dimension one, because $\text{rank}R = 2$, and the system is controllable, i.e. the unreachable modes are at zero.

Question 4

a) Note that, if $w(t) = [\sin t, \cos t, 0]'$ then $\dot{w}(t) = [\cos t, -\sin t, 0]'$ and

$$\begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix}.$$

This implies that $w(t)$ belongs to the class of signals for which asymptotic tracking is achievable.

b) Let $e = x - w$ and note that

$$\dot{e} = \dot{x} - \dot{w} = Ax + Bu - \dot{w} = Ax + Bu - Aw = Ae + Bu.$$

Let now

$$u = Ke = K(x - w)$$

and select $K = [k_1, k_2, k_3]$ such that the eigenvalues of $A + BK$ are all equal to -1. Note that

$$A + BK = \begin{bmatrix} k_1 & 1 + k_2 & 1 + k_3 \\ -1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

The characteristic polynomial of $A + BK$ is

$$p(s) = (s + 1)(s^2 - k_1 s + 1 + k_2)$$

and this should be equal to $(s+1)^3$. This is achieved by selecting $k_1 = -2$ and $k_2 = 0$, while k_3 can be arbitrarily assigned. This is due to the fact that the pair (A, B) is not reachable, however it is stabilizable and the unreachable mode is $s = -1$. This can be seen considering the reachability pencil

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & -1 & -1 & 1 \\ 1 & s & -2 & 0 \\ 0 & 0 & s+1 & 0 \end{array}\right]$$

and noting that it has rank 2 for $s = -1$ and rank 3 for any other s.

Question 5

a) The reachability matrix is

$$R = \begin{bmatrix} 1 & 2 \\ 1 & \alpha+1 \end{bmatrix}$$

and the system is reachable if $\alpha \neq 1$. If $\alpha = 1$ the reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s-1 & -1 & 1 \\ -1 & s-1 & 1 \end{array}\right]$$

and this has rank 1 for $s = 0$ and rank 2 for any other s. Therefore, the system is controllable and stabilizable.

b) The observability matrix is

$$O = \begin{bmatrix} 1 & -1 \\ 1-\alpha & 0 \end{bmatrix}$$

and the system is observable if $\alpha \neq 1$. If $\alpha = 1$ the observability pencil is

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s-1 & -1 \\ -1 & s-1 \\ \hline 1 & -1 \end{array}\right]$$

and this has rank 1 for $s = 2$ and rank 2 for any other s. Therefore the system is not detectable.

c) Let $K = [k_1\ k_2]$ and note that

$$A + BK = \begin{bmatrix} k_1+1 & 1+k_2 \\ \alpha+k_1 & k_2+1 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is

$$s^2 + (-2 - k_1 - k_2)s + (k_2 - \alpha + 1 - k_2\alpha).$$

Hence the selection

$$k_1 = -1 \qquad k_2 = -1$$

is such that the eigenvalues of $A + BK$ are equal to 0. Let $L = [l_1\ l_2]^T$ and note that

$$A + LC = \begin{bmatrix} l_1+1 & 1-l_1 \\ \alpha+l_2 & 1-l_2 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is

$$s^2 + (-2 - l_1 + l_2)s + (-2l_2 + l_1 + 1 + l_1\alpha - \alpha)$$

Hence the selection

$$l_1 = \frac{3+\alpha}{\alpha-1} \qquad l_2 = \frac{1+3\alpha}{\alpha-1}$$

is such that the eigenvalues of $A + LC$ are equal to 0. Finally, the controller is $\dot{\xi} = (A + BK + LC)\xi - Ly$, $u = K\xi$.

d) The limit for $\alpha \to 1$ of $\|K\| = \sqrt{k_1^2 + k_2^2}$ is $\sqrt{2}$, whereas the limit for $\alpha \to 1$ of $\|L\|$ is equal to $+\infty$. This is in agreement with the fact that, for $\alpha = 1$, the system is not reachable but stabilizable, with a non-reachable mode equal to 0, but not detectable.

Question 6

a) Let i_1 and i_2 be the currents through L and C, respectively. Then $y = i = i_1 + i_2$. Moreover,

$$i_1 = x_2 = \frac{u - L\dot{x}_2}{R} \qquad\qquad i_2 = C\dot{x}_1 = \frac{u - x_1}{R}.$$

Hence,

$$A = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} \qquad C = \begin{bmatrix} -\frac{1}{R} & 1 \end{bmatrix} \qquad D = \frac{1}{R}.$$

b) The reachability matrix is

$$\mathcal{C} = \begin{bmatrix} \frac{1}{RC} & -\frac{1}{R^2C^2} \\ \frac{1}{L} & -\frac{R}{L^2} \end{bmatrix}$$

and it is full rank if $R^2C \neq L$. If $R^2C = L$ the system is not reachable. However, as both the eigenvalues of A have negative real part, the system is stabilizable.

c) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} -\frac{1}{R} & 1 \\ \frac{1}{R^2C} & -\frac{R}{L} \end{bmatrix}$$

and it is full rank if $R^2C \neq L$. If $R^2C = L$ the system is not observable. However, as both the eigenvalues of A have negative real part, the system is detectable.

d) Consider new coordinates $\hat{x}$ such that $x = L\hat{x}$ and L is constructed from the first column of the reachability matrix, multiplied by R^2C, and a second column which renders L invertible:

$$L = \begin{bmatrix} R & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\dot{\hat{x}} = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

These equations show that the unreachable subsystem, namely

$$\dot{\hat{x}}_2 = -\frac{1}{RC}\hat{x}_2$$

is observable, and that the unobservable subsystem, namely

$$\dot{\hat{x}}_1 = -\frac{1}{RC}\hat{x}_1 + u$$

is reachable. Hence, there is no subsystem which is both reachable and observable.

e) The input-output behaviour of the first electrical network is described by

$$Ce^{At}B + D = \frac{e^{-\frac{R}{L}t}}{L} - \frac{e^{-\frac{1}{RC}t}}{R^2C} + 1/R.$$

This, for $L = R^2C$, is equal to $1/R$, which describes the input-output behaviour of the second circuit.