Master

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

EEE/ISE PART III/IV: M.Eng. and ACGI

CONTROL ENGINEERING

Time allowed: 3:00 hours

There are SIX questions on this paper.
Answer FOUR questions.

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker: Astolfi, A.

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1. Figure 1 shows the block diagram of an idealized aircraft pitch control system, with variable forward path gain K, angle, velocity and acceleration feedback. Show that, as far as stability properties are concerned, the systems is equivalent to that with block diagram given in Figure 1(b), in which

$$G(s) = \frac{1}{s^3}.$$

(a) What is the transfer function $H(s)$? [2]

(b) Sketch the extended Nyquist diagram of $G(s)H(s)$. Determine the intercept with the real axis. [14]

(c) Describe how the stability of the closed loop system changes, as K increases in the range

$$0 \leq K < \infty.$$

[4]

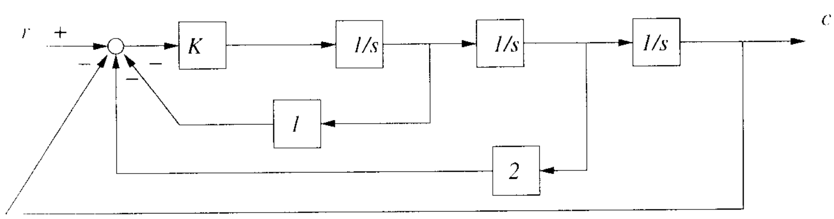

Figure 1(a)

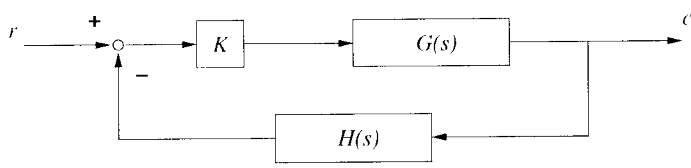

Figure 1(b)

2. Consider the mass spring system of Figure 2. Two unit masses are attached, at one end, to rigid supports via springs, each with unit spring constant. They are attached to each other by a dashpot, with unit gain, *i.e.* the tension T across the dashpot is related to the relative velocity v of the masses away from each other, according to

$$T = v .$$

The output y is provided by a strain gauge on the dashpot that measures tension:

$$y = T .$$

(a) Derive a state space model of the (control free) system with output y, of the form

$$\begin{cases} \dot{x} = Ax \\ y = c^T x , \end{cases}$$

taking as state variables x_1 and x_3 the displacements of the left and right masses, and taking as state variables x_2 and x_4 the velocities of the left and right masses, respectively. [12]

(b) Show that the system is not observable. [6]

(c) Find a non-zero value of the initial state vector

$$x(0) = (x_1, \ldots, x_4)^T$$

which is 'non-observable', in the sense that, if $y(t)$ is the corresponding output, then

$$y(t) = \quad \text{for all } t \geq 0 .$$

[2]

Note: in the last part of the question, you should use physical reasoning, concerning the nature of oscillations such that $T(t) = 0$, for all $t \geq 0$; no detailed calculations are required.

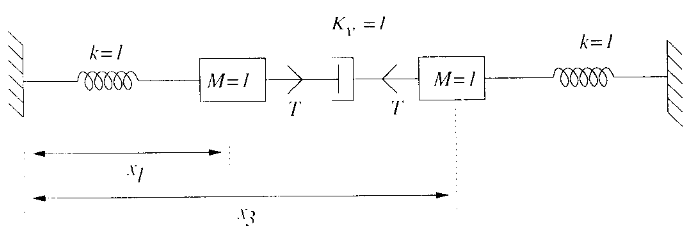

Figure 2

3. Figure 3 illustrates the block diagram of a ship stabilizer system, incorporating velocity feedback and phase lag compensation. Here

$$D(s) = K \times \frac{1+T_1 s}{1+T_2 s}, \quad H(s) = (1+T_D) \quad \text{and} \quad G(s) = \frac{1}{s(1+0.6s)^2}.$$

(The positive constants K, T_1, T_2 and T_D are design parameters with $T_2 > T_1$.)

(a) Show that Laplace transform of the error signal $e = r - c$ (the difference between the reference input r and the output c) is related to the Laplace transform of the reference signal r according to the following formula:

$$e(s) \;=\; \left(\frac{1}{1+D(s)G(s)H(s)} \;+\; T_D \times \frac{sD(s)G(s)}{1+D(s)G(s)H(s)}\right) r(s)\,.$$

Hence, or otherwise, show that the steady state error for a unit ramp input r(t) is

$$\lim_{t\to\infty}(r(t)-c(t)) \;=\; \left(\frac{1}{K}+T_D\right)\,.$$

[6]

Note that the error signal is not s (see Figure 3) because the system does not have unity feedback.

(b) Choose values of K, T_1, T_2 and T_D to achieve the following specifications:

(i) The phase margin is 60°.

(ii) The gain cross over frequency is $\bar{\omega} = 1\,rad\ sec^{-1}$.

(iii) The steady state error for a unit ramp input is $0.6\ rad\ sec^{-1}$. [14]

In part (b) you should use the following procedure:

Step 1: Assuming that $\frac{1}{T_1}$, $\frac{1}{T_1} << \bar{\omega}$, choose T_D to achieve the phase margin and bandwidth specifications (i) and (ii). fi

Step 2: Choose K to satisfy the steady state error specication (iii).

Step 3: Assuming $\frac{1}{T_1} = 0.1\,\bar{\omega}$, choose T_1 and T_2.

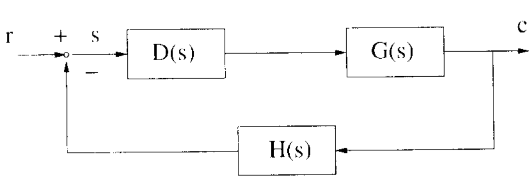

Figure 3

1. The angular displacement θ of a satellite is related to the torque T applied to the control actuator according to:

$$d^2\theta/dt^2 \;=\; T\,.$$

The actuator torque is related to a control signal u by a first order lag:

$$5T + dT/dt \;=\; u\,.$$

(a) Derive a state space model, governing these variables, in which the state components are:

$$x_1 = \theta, \quad x_2 = d\theta/dt \quad \text{and} \quad x_3 = T\,.$$

[2]

(b) Design a position + velocity feedback control

$$u \;=\; -k_1\,\theta \;-\; k_2\,d\theta/dt\,, \qquad (1)$$

to locate two closed loop poles at

$$s \;=\; -1 + j \;\text{ and }\; -1 - j\,.$$

[14]

(c) Where is the third closed loop pole located? [2]

(d) Briefly discuss whether we are justified in interpreting the two poles (1) as 'dominant' poles. [2]

5. The displacement of a mechanical system satisfies the equation

$$d^2y/dt^2 = u,$$

in which $u(t)$ is the applied force. A proportional + derivative control law

$$u = -k_1 y - k_2 dy/dt$$

is required, that minimizes the cost function

$$\int_0^\infty \left(\alpha y^2(t) + u^2(t)\right) dt$$

(for fixed initial values of y and dy/dt). Here α is a given constant.

(a) Redefine the design problem as an optimal control problem

$$(Q) \begin{cases} \text{Minimize } \int_0^\infty \left(x^T(t)Qx(t) + u^2(t)\right) dt \\ \text{subject to } dx/dt = Ax + \mathbf{b}u \\ x(0) = x_0 \end{cases}$$

(for fixed x_0), by selecting appropriate values of the matrices

$$A\ (2 \times 2), \quad Q\ (2 \times 2) \text{ and } \mathbf{b}\ (2 \times 1).$$

[2]

(b) Hence find the optimal gain

$$k^T = [k_1 \quad k_2].$$

[12]

(c) Derive the closed loop differential equation satified by $y(t)$ for general α. [2]

(d) Describe how the damping factor ζ and the undamped natural frequency ω_n of the output $y(t)$ vary as $\alpha \to \infty$, for closed loop operation. [4]

You can use the fact that the solution to (Q) is given by

$$u = -\mathbf{b}^T P\, \mathbf{x},$$

where P is a solution of the Algebraic Riccati Equation (ARE):

$$\begin{cases} A^T P + PA + Q - P\mathbf{b}\mathbf{b}^T P = 0. \\ P = P^T \quad \text{and} \quad P > 0. \end{cases}$$

Note

(i): For problem (OC), we can arrange that the condition '$P > 0$' is satisfied by choosing 'positive square roots', when solving the equations arising from (ARE).

(ii): A second order system with damping factor zeta and undamped natural frequency ω_n has transfer function with denominator

$$s^2 + 2\zeta\omega_n s + \omega_n^2.$$

6a. Consider the nonlinear 'relay with hysteresis' device with input/output characteristic indicated in Figure 6(a), in which $\epsilon\,(>0)$ and $V\,(>0)$ are positive constants. The diagram shows the square wave output $n(t)$ (of amplitude V), for a sinusoidal input $e(t) = Asin(\omega t)$, when $A > \epsilon$. Notice that the output $n(t)$ does not switch from $+V$ to $-V$ until $e(t) < -\epsilon$ and does not switch from $-V$ to $+V$ until $e(t) > +\epsilon$.

(i) Show that the (complex) describing function of the device is

$$N(A) = \frac{4V}{\pi A}\exp\{-j\sin(\epsilon/A)\}\,.$$

[6]

(ii) Sketch the locus of $-\frac{1}{N(A)}$ for A in the range $\epsilon < A < \infty$. [2]

Hint: For fixed A and a sinusoidal input $e(t) = Asin(\omega t)$, interpret the output as the output from a pure relay in series with a time delay (that depends on A).

6b. Figure 6(b) shows the block diagram of a velocity control system, incorporating, in the forward path, a device with transfer function $\frac{1}{s(1+s)^2}$ and an amplifier (NL) that has failed. Due to this failure, the amplifier behaves like a relay with hysteresis, as in part (a), in which

$$\epsilon = 0.5 \quad rad\ s^{-1}$$

and V is an unknown parameter.

A limit cycle is observed with amplitude $A = 1\,rad\ s^{-1}$ (at the output).

(i) What is the frequence of the limit cycle oscillation? [10]

(ii) Assess the stability of the limit cycle. [2]

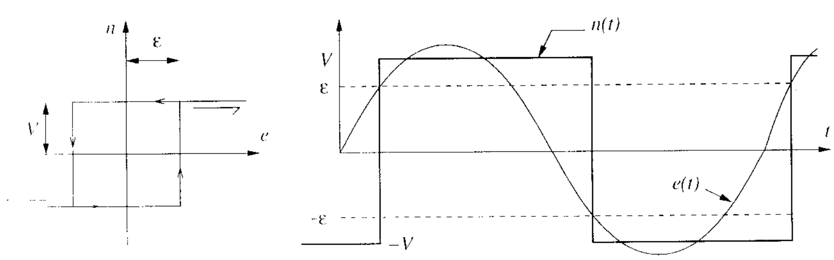

Figure 6(a)

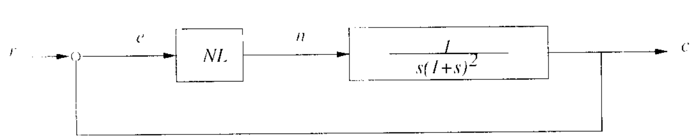

Figure 6(b)

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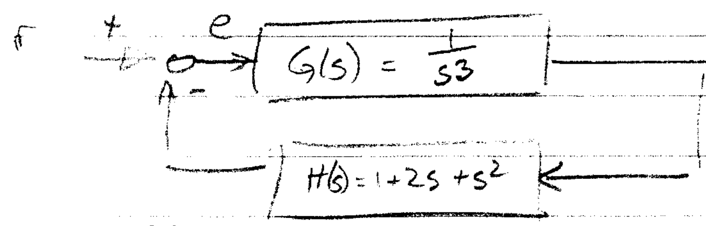
1. $e = r - c - 2sc - s^2 c = r - H(s)c$ where

$H(s) = 1 + 2s + s^2$.

and $c = \frac{1}{s^3} \cdot e = G(s)$ where $G(s) = \frac{1}{s^3}$

The block diagram can then be reorganized as



$GH = \frac{1+2s+s^2}{s^3}$

[2]

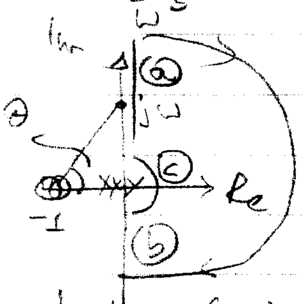
Write $\tilde{G}(s) = G(s)H(s)$.

Nyquist diagram of $\tilde{G}(s)$ intercepts real axis when $\omega = \bar{\omega}$, where

$$\tilde{G}(j\bar{\omega}) = \frac{1 - \bar{\omega}^2 + 2j\bar{\omega}}{-j\bar{\omega}^3} = j\frac{1-\bar{\omega}^2}{\bar{\omega}^3} - \frac{2}{\bar{\omega}^2} = d + 0j.$$

Then intercept is $-d$. Equating Re + Im parts gives

$\frac{1-\bar{\omega}^2}{\bar{\omega}^3} = 0 \Rightarrow \bar{\omega} = 1$. Then $d = -2$ ← intercept.



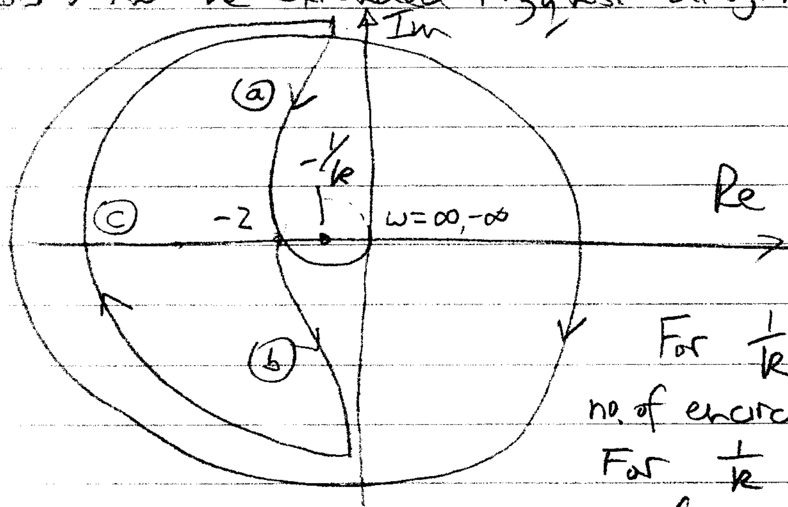
location of Φ poles/zeros.

$$\measuredangle \tilde{G}(j\omega) = 2\theta - 3 \times 90^\circ. \text{ So}$$

$$-3 \times 90^\circ \le \measuredangle \tilde{G}(j\omega) \le -1 \times 90^\circ$$

$$\underset{\omega=0}{\infty} \ge |G(j\omega)| \ge \underset{\omega=\infty}{0}$$

[14] We can now draw the extended Nyquist diagram:



For $\frac{1}{K} < 2$,
no. of encirclements = 0

For $\frac{1}{K} > 2$
no. of encirclements = 2

So the system is unstable for $0 < K < \frac{1}{2}$

[4] " " " stable for $\frac{1}{2} < K$

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2. Equations of motion are

$\ddot{x}_1 = T - x_1$, $\ddot{x}_3 = -T - x_3$, $T = \dot{x}_3 - \dot{x}_1$, $y = T = \dot{x}_3 - \dot{x}_1$.

Let $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3$.

Then

$$\dot{x}_2 = T - x_1 = x_4 - x_2 - x_1$$
$$\dot{x}_4 = -T - x_3 = x_2 - x_4 - x_3$$

the state space equations for $x = [x_1, \ldots, x_4]^T$ and y are

$$\begin{cases} \dot{x} = Ax \\ y = c^T x \end{cases} \quad \text{with} \quad A = \begin{bmatrix} 0 & +1 & 0 & 0 \\ -1 & -1 & 0 & +1 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & -1 & -1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ -1 \\ 0 \\ +1 \end{bmatrix}$$

[12]

Observability matrix is $M =$

$$\begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & +1 \\ +1 & 2 & -1 & -2 \\ -2 & -3 & +2 & +3 \\ 3 & 4 & -3 & -4 \end{bmatrix}$$

Notice that the 3rd column is a multiple of the first column. It follows

$$\det M = 0$$

[6] This tells us that the system is not observable

If the initial state is

$x(0)^T = [\alpha \quad 0 \quad \alpha \quad 0]^T$ (for any number α)

Then the masses oscillate in such a way that the distance between them stays constant. Since the output $y(t)$ is the rate of change of this distance,

[2] $$y(t) = 0 \quad \text{for all } t \geq 0$$

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3 (b) $e(s) = r(s) - \dfrac{DG}{1+DG(1+T_D s)} r(s) = \left[1 - \dfrac{DG}{1+DG(1+T_D s)}\right] r(s)$

$$= \left[\frac{1}{1+DG(1+T_D s)} + T_D\, s \frac{DG}{1+DG(1+T_D s)}\right] r(s)$$

Then, if $r(s) = 1/s^2$ (unit ramp)

$$\text{s.s. error} = \lim_{s \to 0} s e(s) = s\left(\frac{s}{s + \frac{K(1+T_1 s)}{(1+T_2 s)} \cdot \frac{(1+T_D s)}{(0.1s+1)^2}} + T_D s \frac{\frac{K(1+T_1 s)}{(1+T_2 s)} \cdot \frac{1}{(0.1s+1)^2}}{s + \frac{K(1+T_1 s)(1+T_D s)}{(1+T_2 s)(0.1s+1)^2}}\right) \times \frac{1}{s^2}$$

[6] $$= \frac{1}{0+K} + \frac{T_D K}{0+K} = \frac{1}{K} + T_D$$

Assume $\frac{1}{T_1}, \frac{1}{T_2} << \bar{\omega} = 1$ (gain cross-over frequency)

Then $\measuredangle DG(j\bar{\omega}) = -90^0 - 2 \times \tan^{-1} 0.6 = -151.9275^0$

We must have $\measuredangle DG(j\bar{\omega})(1+T_D j\bar{\omega}) = -180^0 + 60^0$ (60^0 phase margin)

Hence $\measuredangle(1+T_D j\bar{\omega}) = 31.9275$, so $T_D = \tan(31.9275) \Rightarrow T_D = 0.623$

We require however

$$\frac{1}{K} + T_D = 0.75$$

Hence $\frac{1}{K} = 0.127$ Hence $K = 7.874$

Finally we must choose T_1 and T_2 to ensure $DG(1+T_D s)$ has unity gain at gain cross-over frequency, i.e.

$$K \times \frac{T_1}{T_2} \times |G(j\bar{\omega})| \times |1 + T_D j\bar{\omega}| = 1$$

i.e. $$7.874 \times \left(\frac{T_1}{T_2}\right) \times \frac{1}{(1+0.6^2)} \times \sqrt{1 + 0.623^2} = 1$$

$$\Rightarrow \quad 7.874 \times \left(\frac{T_1}{T_2}\right) \times \frac{1}{1.36} \times 1.17818 = 1$$

So $$\frac{T_1}{T_2} = \frac{1.36}{1.17818 \times 7.874} = 0.146599284$$

Take $\frac{1}{T_1} = 0.1\bar{\omega} \Rightarrow T_1 = 10$ sec

So

$$T_2 = 68.21 \text{ sec}$$

[14] Compensation: $D(s) = 7.874 \times \dfrac{1+10s}{1+68.21s}$ and $H(s) = 1 + 0.623s$

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4. The state variables are $x_1 = \theta$, $x_2 = \dot{\theta}$ and $x_3 = T$. We have

$\dot{x}_1 = \dot{\theta} = x_2$, $\dot{x}_2 = \ddot{\theta} = T = x_3$ and $\dot{x}_3 = \dot{T} = -5T + u = -5x_3 + u$

The state space model is therefore

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \qquad (= Ax + bu)$$

We must design feedback of the form

$$u = -k_1\theta - k_2\dot{\theta} = -k_1x_1 - k_2x_2 - 0.x_3$$

to give the appropriate closed loop characteristic polynomial.

But det $[sI - A - bk^T] = k_1 + k_2 s + 5s^2 + s^3$.

(This follows from the 'companion-form' structure of A).

Match this to

$$(s+1+j)(s+1-j)(s+\alpha) = (s^2 + 2s + 2)(s+\alpha)$$
$$= s^3 + (\alpha+2)s^2 + (2\alpha+2)s + 2\alpha$$

for some α. We have

$$k_1 = 2\alpha, \quad k_2 = 2\alpha + 2, \quad 5 = \alpha + 2$$

Solving this equations gives

$$\alpha = 3, \; k_1 = 6 \text{ and } k_2 = 8.$$

The required proportional + velocity feedback is therefore

$$u = -6\theta - 8\dot{\theta}$$

This places two closed loop poles at $s = -1 \pm j$. The remaining closed loop pole is at

$$s = -3.$$

The '3rd' pole is a non-oscillatory pole with decay rate 3 sec^{-1}. The 'undamped' natural frequency of the oscillatory poles is 1 sec^{-1}. It follows that transients associated with the $s=-3$ pole will decay rapidly, compared with those associated with the oscillatory poles. The oscillatory poles can therefore be regarded as 'dominant'

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5. Introduce the state vector components $x_1 = y$ and $x_2 = \dot{y}$. Then $\dot{x}_2 = u$ and $\dot{x}_1 = x_2$. The dynamic equations can then be expressed:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The cost is $\int_0^\infty (\alpha |y(t)|^2 + |u(t)|^2)\,dt = \int_0^\infty \left(x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + u^2(t)\right)dt$.

The design problem can therefore be expressed as (Q), with

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $Q = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$

[2]

Write $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ for a solution to ARE. Then

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = 0$$

or

$$\begin{bmatrix} 0 & 0 \\ P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ 0 & P_{12} \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{12}^2 & P_{12}P_{22} \\ P_{12}P_{22} & P_{22}^2 \end{bmatrix} = 0$$

Equating coefficients gives

$\alpha = P_{12}^2$, $P_{11} = P_{12}P_{22}$ and $2P_{12} = P_{22}^2$

Taking positive sq. roots gives $P_{12} = \alpha^{\frac{1}{2}}$ and $P_{22} = 2^{\frac{1}{2}}\alpha^{\frac{1}{4}}$

Hence the optimal feedback is

[12] $u = -b^T P x = -[P_{12} \;\; P_{22}]\,x = -\alpha^{\frac{1}{2}} x_1 - 2^{\frac{1}{2}}\alpha^{\frac{1}{4}} x_2$

The differential equation for y is

[2] $\ddot{y}(t) + \sqrt{2}\,\alpha^{\frac{1}{4}}\dot{y}(t) + \alpha^{\frac{1}{2}} y(t) = 0$

Match to $\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = 0$.

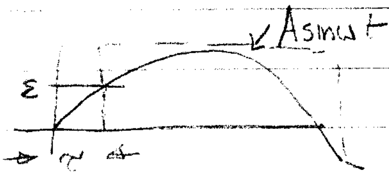
We get

$\omega_n = \alpha^{\frac{1}{4}}$ and $2\xi\alpha^{\frac{1}{4}} = \sqrt{2}\,\alpha^{\frac{1}{4}}$ or $\xi = 1/\sqrt{2}$

[4] We see that the undamped natural frequency $\omega_n \to \infty$ as $\alpha \to \infty$ and the damping factor $\xi = \frac{1}{\sqrt{2}}$ for all α

6 Apply an input $e(t) = A\sin\omega t$

N/L $\equiv$ $A\sin\omega t \to$ pure relay $\xrightarrow{\frac{\pi V}{4A}\sin\omega t + \dots}$ delay $\to$



Hysteresis introduces a time delay of τ secs where

$A\sin\omega\tau = \varepsilon$ or $\tau = \frac{1}{\omega}\sin^{-1}\left(\frac{\varepsilon}{A}\right)$

Ignoring time delay, first harmonic has amplitude:

$$\frac{2}{T}\int_0^T m(t)\sin\omega t\,dt = \frac{4V}{T}\int_0^{T/2}\sin\omega t\,dt = \frac{4V}{\omega T}\left(\cos(0) - \cos\left(\frac{\omega T}{2}\right)\right)$$

But $\omega T = 2\pi$, whence first harmonic has amplitude $\frac{4V}{2\pi}\times 2 = \frac{4V}{\pi}$

So first harmonic is $\frac{4V}{\pi}\sin(\omega(t-\tau)) = \frac{4V}{\pi}\sin\left(\omega t - \underbrace{\sin^{-1}\left(\frac{\varepsilon}{A}\right)}_{\phi}\right)$

Nonlinearity behaves like a linear system with "complex" gain

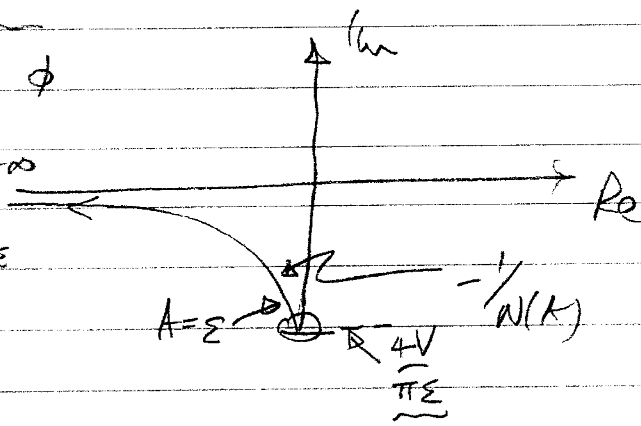
[6] $N(A) = \frac{4V}{\pi A} e^{-j\sin^{-1}(\varepsilon/A)}$. (Describing Function)

'gain' phase ϕ

$|N(A)| \to \infty$ as $A \to \infty$

[2] $\measuredangle -\frac{1}{N(A)} = +\sin^{-1}\left(\frac{\varepsilon}{A}\right) + 180^\circ$

$= \begin{cases} 270^\circ \equiv +90^\circ, & A = \varepsilon \\ 0^\circ & A = \infty \end{cases}$



Limit cycle equation is

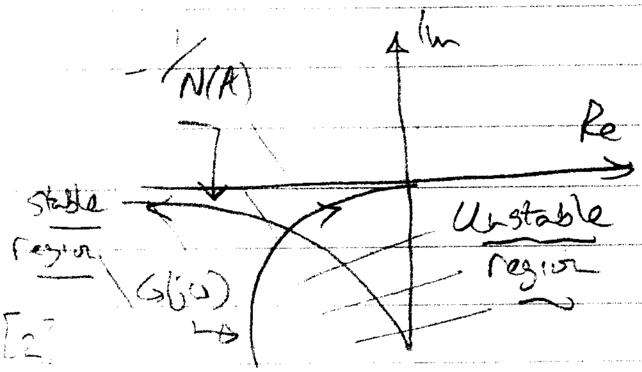
$$G(j\bar{\omega}) = -\frac{1}{N(A)} = -\frac{\pi A}{4V}\exp\left\{j\sin^{-1}\left(\frac{\varepsilon}{A}\right)\right\}$$

We know $\varepsilon = \frac{1}{2}$ and $A = 1$ so

$$\frac{1}{G(j\bar{\omega})} = j\bar{\omega}(1+j\bar{\omega})^2 = -\frac{4V}{\pi}\exp\{\overbrace{j\sin^{-1}\tfrac{1}{2}}^{30^\circ}\}$$

$\measuredangle\frac{1}{G(j\bar{\omega})} = 90^\circ + 2\tan^{-1}\bar{\omega} = 180^\circ + 30^\circ \Rightarrow \bar{\omega} = \tan 60^\circ = \sqrt{3}$

[10] Also, $\left|\frac{1}{G(j\bar{\omega})}\right| = \sqrt{3}(1+3) = \frac{4V}{\pi}$. Hence $V = \frac{\pi\sqrt{3}}{16}$



As A increases $-\frac{1}{N(A)}$ moves from unstable to stable region.

Hence

Limit Cycle is stable

[2]