Paper Number(s): **E3.09**
ISE3.9

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2001

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Tuesday, 8 May 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

Time allowed: 3:00 hours

Examiners: Vinter,R.B. and Astolfi,A.

Corrected Copy

Special instructions for invigilators: None

Information for candidates: None

1. State Nyquist's Theorem relating the Nyquist diagram of the forward path transfer function of a unity feedback control system to the number of 'unstable' open loop and closed loop poles of the system.

Consider the closed loop control system of *Figure 1(a)*, in which

$$G(s) = \frac{1}{(s-1)^2}.$$

Here, k is an adjustable parameter (the 'velocity feedback gain'). Investigate the effects on system stability of increasing k, using Nyquist's Theorem.

You should use the following method.

(i) Show that the closed loop poles of the system of Figure 1(a) coincide with the closed loop poles of the unity feedback system of *Figure 1(b)*, in which

$$\tilde{G} = \frac{sG(s)}{1+G(s)}.$$

(ii) Sketch the Nyquist Diagram of $\tilde{G}(s)$. (You are required to calculate the intercepts with the real axis.)

(iii) By interpreting the Nyquist diagram of $\tilde{G}(s)$, describe how the closed loop stability properties of the system of *Figure 1(a)* are affected, as k increases over the range $0 \le k < \infty$.

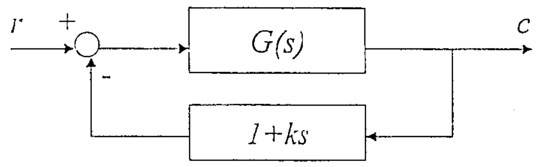


Figure 1(a)

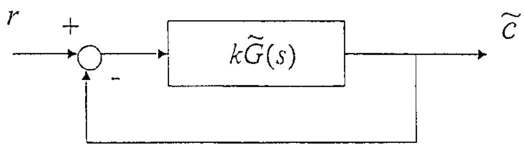


Figure 1(b)

2. *Figure 2* illustrates a cart of mass M, attached to a rigid support by a spring (spring constant K). The cart carries a mechanical accelerometer, comprising a mass m, spring (spring constant k) and a damper (damper constant K_d) .

The absolute displacement of the cart is z. The displacement of the accelerometer relative to the cart is y.

Show that z and y are governed by the equations

$$
\begin{aligned}
d^2z/dt^2 &= -(\frac{K}{M})z + (\frac{k}{M})y + \frac{K_d}{M}dy/dt \\
d^2y/dt^2 &= -k(\frac{1}{m} + \frac{1}{M})y - K_d(\frac{1}{m} + \frac{1}{M})dy/dt + (\frac{K}{M})z \,.
\end{aligned}
$$

Derive (control free) state space equations,

$$dx/dt = Ax \quad \text{and} \quad y = c^T x \,, \tag{1}$$

for the system, with state vector $x = (z, dz/dt, y, dy/dt)$ and scalar output y. Show that (1) is *not* observable, if $K = 0$. Why? (10 25)

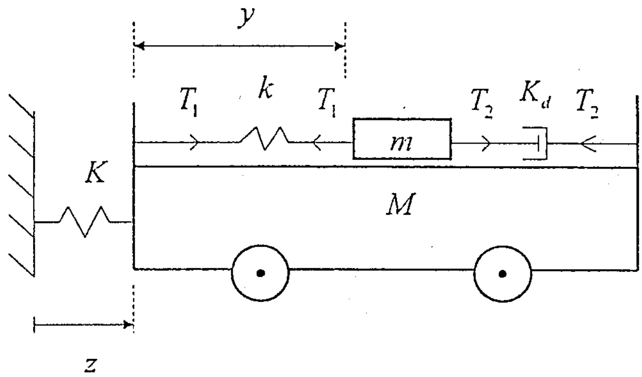


Figure 2

3. Sketch the amplitude and phase frequency response of a phase lag compensator $D_{lag}(s)$ and of a phase advance compensator $D_{adv}(s)$:

$$D_{lag}(s) = \frac{1+s/\omega_1}{1+s/\omega_0}, \quad \omega_0 < \omega_1, \quad \text{and} \quad D_{adv}(s) = \frac{1+s/\omega_2}{1+s/\omega_3}, \quad \omega_2 < \omega_3 .$$

Explain why there is a practical design limitation on the size of ω_3/ω_2.

Consider the control system of *Figure 3*, in which

$$G(s) = \frac{2}{s(s+1)^2} .$$

Design a lag lead compensator

$$D(s) = D_{lag}(s) D_{adv}(s)$$

(with $D_{lag}(s), D_{adv}(s)$ as above), to achieve the following specifications for the compensated system.

(a) $\omega_c = 0.9\ rads^{-1}$,

(b) $\phi = 60°$,

(c) $\omega_3/\omega_2 \leq$ 10

where ω_c is the gain crossover frequency, i.e. the frequency ω_c such that

$$|D(j\omega_c)G(j\omega_c)| = 1 ,$$

and ϕ is the phase margin.

To carry out your design, you should use the following steps.

Step 1. Design phase advance compensation $D_{adv}(s)$ such that $\omega_c = \omega_{max}$ and $\angle D_{adv}(j\omega_c)G(j\omega_c)$ gives a phase margin of $60°$. Check (c).

Step 2. Choose the phase lag compensation $D_{lag}(s)$ such that

$$|D_{lag}(j\omega_c) D_{adv}(j\omega_c) G(j\omega_c)| = 1 .$$

You can quote the facts that the maximum phase advance of $D_{adv}(j\omega)$ is $90° - 2 \times \tan^{-1}\sqrt{\omega_2/\omega_3}$, and occurs at $\omega = \sqrt{\omega_2\omega_3}$.

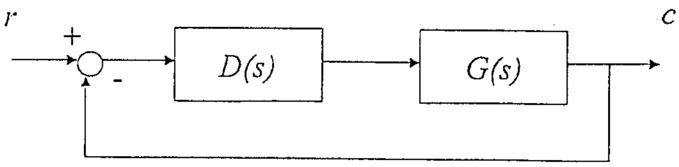


Figure 3

4(a). Consider the unity feedback system of *Figure 4*, in which

$$G(s) = \frac{k}{s+a}.$$

Here, $a > 0$ is a modeling constant and $k > 0$ is a variable gain.
Determine the open loop gain cross-over frequency ω_c of $G(j\omega)$, *i.e.* the frequency ω_c such that $|G(j\omega_c)| = 1$.
Determine also the rise time t_r of the closed loop system, defined by

$$y(t_r) = 0.99 \times y(t = \infty),$$

where $y(t)$ is the unit step response of the closed loop system, initially at rest.
Show that, for all k, ω_c and t_r are related according to

$$\omega_c^2 = \frac{log_e(100)}{t_r} \cdot \left[\frac{log_e(100)}{t_r} - 2a\right].$$

Deduce that, for t_r small,

$$\omega_c t_r = constant.$$

('gain cross-over frequency is inversely proportional to rise time') What is the value of the constant?

4(b). Consider again the unity feedback system of *Figure 4*, for general $G(s)$. ~~Assume K = 1~~. (9 50)

Fix a number $N \geq 0$. Suppose $\bar{\omega}$ is a frequency for which the closed loop phase frequency response satisfies

$$\angle \frac{G(j\bar{\omega})}{1 + G(j\bar{\omega})} = tan^{-1}(N).$$

Show that $G(j\bar{\omega})$ lies on an 'N' circle in the complex plane, namely the set of points with coordinates (X, Y) which satisfy the equation

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \left(\frac{1}{2N}\right)\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2.$$

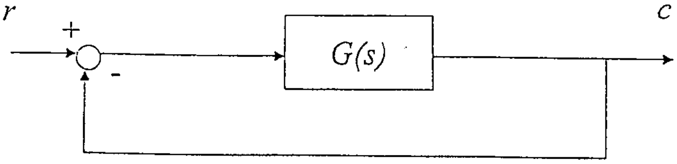


Figure 4

5(a). Consider the dynamic system of *Figure 5(a)*, relating the input u to the output y, in which the transfer function is

$$G(s) = \frac{1}{(s+3)(s-1)s^2}$$

Derive a state space model with states $x_1 = y$, $x_2 = dy/dt$, $x_3 = d^2y/dt^2$, $x_4 = d^3y/dt^3$.

Choose the parameters k_1, k_2 and k_3 in the proportional + velocity + acceleration controller

$$u = -k_1x_1 - k_2x_2 - k_3x_3$$

to arrange that the closed loop characteristic polynomial is of the form

$$(s + \alpha(1+j))^2(s + \alpha(1-j))^2$$

for some $\alpha \geq 0$, i.e. all closed loop eigenvalues have damping factor $1/\sqrt{2}$ and are equidistant from the origin.

5(b). Consider now the control system of *Figure 5(b)* to stabilize the orientation θ of a rocket in the plane. The control system provides proportional + velocity + acceleration control, except that the accelerometer hardware includes a first order lag.

Use the results of part (a) to choose k, k_v and k_a, such that all the closed loop eigenvalues have damping factor $1/\sqrt{2}$ and are equidistant from the origin.

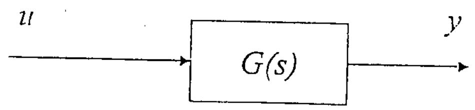
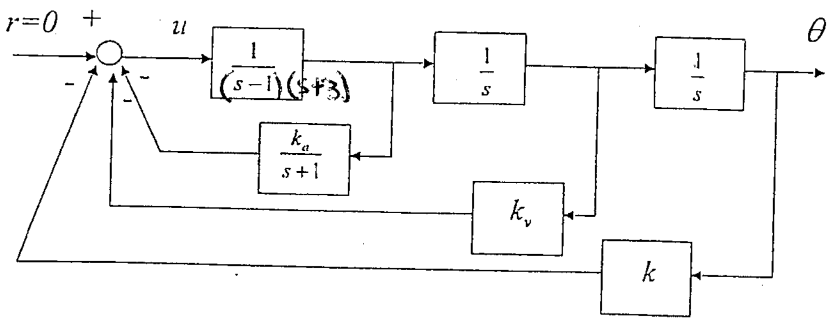


Figure 5(a)



(11 3c)

Figure 5(b)

6. Derive the describing function $N(A)$ of the amplifier with gain K and off-set at the origin a, whose characteristic is shown in *Figure 6(a)*.

   Hint: decompose the nonlinearity ~~is~~ as the sum of a pure gain and an ideal relay. (9 5°)

   Such a device is present in the forward path of the control system of *Figure 6(b)*, in which

$$G(s) = \frac{48}{(s+2)^3}.$$

   Estimate the frequency of limit cycle oscillations, predicted by describing function analysis.

   It is known that $K = 1$. It is observed that the amplitude of limit cycle oscillations of the output $y(t)$ is 0.01 units. Determine the magnitude of the amplifier offset a.

   Assess whether the limit cycle is stable.

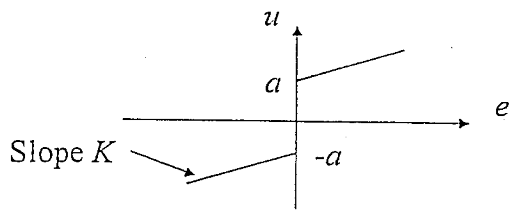


Figure 6(a)

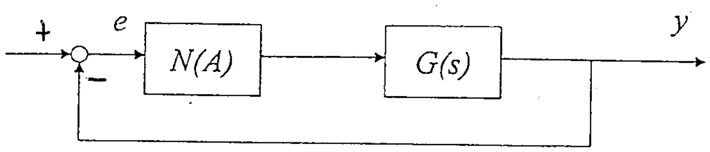


Figure 6(b)

3EE Control Eng, 2001 exam: model answers, E3.09, ISE 3.9 — 27 April 01

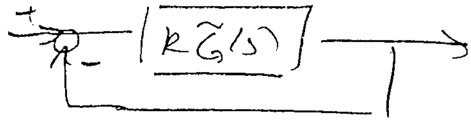
1\. 2 $N = C - O$ where $N = \#$ clockwise encirclements, $C = \#$ ₵ poles, $N = \#$ Ȼ pol[es]

The closed loop poles of 1(a) are the zeros of $1 + (1 + ks)G(s)$.

These coincide with the zeros of $1 + G(s) + ksG(s)$ and therefore

$$1 + k\frac{sG(s)}{1+G(s)} = 1 + k\tilde{G}(s).$$

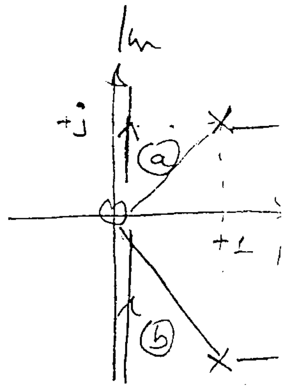
But the zeros of $1 + k\tilde{G}(s)$ are the closed loop poles of:

4 

$(\tilde{G} = sG/(1+G))$

$$\tilde{G}(s) = \frac{s/(s-1)^2}{1 + 1/(s-1)^2} = \frac{s}{s^2 - 2s + 2}.$$

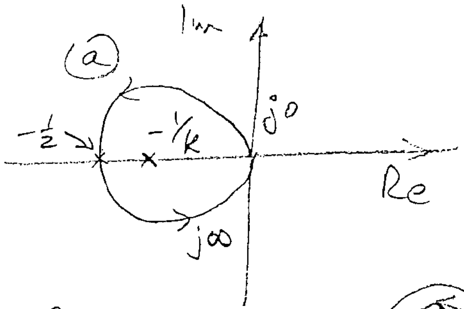
$$\tilde{G}(j\omega) = \frac{j\omega}{-\omega^2 - 2j\omega + 2} \quad \text{is real when } \omega = \pm\sqrt{2}.$$

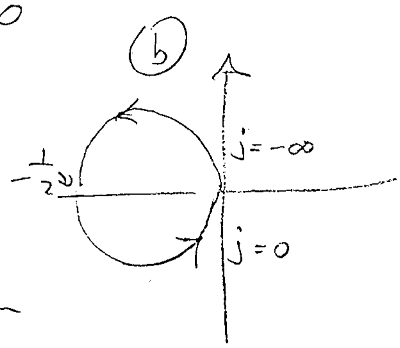


Then $\tilde{G}(j(\omega = \sqrt{2})) = -\frac{1}{2}$

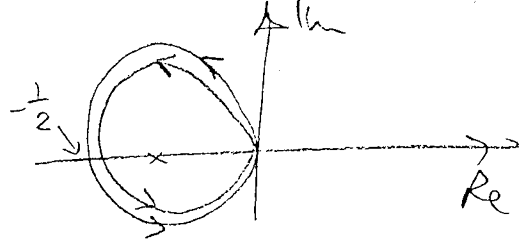
For $j\omega$ on path segment (a), we have

	$\angle \tilde{G}(j\omega)$	$\lvert\tilde{G}(j\omega)\rvert$
$\omega = 0$	$+90^\circ - (360^\circ)$	0
$\omega = \sqrt{2}$	$+90^\circ - 270^\circ$	$-\frac{1}{2}$
$\omega = +\infty$	$+90^\circ - 180^\circ$	0





Overall:

10 

goes round twice

For $k > 2$

(encirclements) $N = -2$, # open loop poles $O = 2$

So # closed loop poles $= C = N + O = 0$

Stable for $k > 2$

For $k < 2$

$N = 0$, $O = 2$, $C = O = 2$

4 Unstable (2 unstable poles) for $k < 2$.

2. Motion of cart: $M\ddot{z} = -Kz + T_1 - T_2$

Motion of load: $m(\ddot{z}+\ddot{y}) = T_2 - T_1$

Spring and damper: $T_1 = ky$ and $T_2 = -K_d\dot{y}$.

Hence

$$\begin{cases} \ddot{z} = -\frac{K}{M}z + \frac{k}{M}y + \frac{K_d}{M}\dot{y} \\ m\ddot{y} = -m\left[-\frac{K}{M}z + \frac{k}{M}y + \frac{K_d}{M}\dot{y}\right] - K_d\dot{y} - ky. \end{cases}$$

$\Longrightarrow$

12 $\underline{\ddot{z} = -\left(\frac{K}{M}\right)z + \left(\frac{k}{M}\right)y + \left(\frac{K_d}{M}\right)\dot{y}}$ and $\underline{\ddot{y} = -k\left(\frac{1}{M}+\frac{1}{m}\right)y - K_d\left(\frac{1}{M}+\frac{1}{m}\right)\dot{y} + \left(\frac{K}{M}\right)z}$

Set $x_1 = z$, $x_2 = \dot{z}$, $x_3 = y$, $x_4 = \dot{y}$. Then

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{K}{M}\right) & 0 & \left(\frac{k}{M}\right) & \left(\frac{K_d}{M}\right) \\ 0 & 0 & 0 & 1 \\ +\frac{K}{M} & 0 & -k\left(\frac{1}{M}+\frac{1}{m}\right) & -K_d\left(\frac{1}{M}+\frac{1}{m}\right) \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and

4 $$y = \underbrace{[0 \quad 0 \quad 0 \quad 1]}_{c^T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The observability matrix is

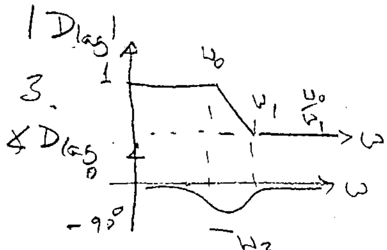
$$\begin{bmatrix} \leftarrow c^T \rightarrow \\ \leftarrow c^T A \rightarrow \\ \vdots \\ \leftarrow c^T A^3 \rightarrow \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ +\left(\frac{K}{M}\right) & 0 & -k\left(\frac{1}{M}+\frac{1}{m}\right) & -K_d\left(\frac{1}{M}+\frac{1}{m}\right) \\ -\frac{K_d K}{M}\left(\frac{1}{M}+\frac{1}{m}\right) & K/M & +kK_d\left(\frac{1}{M}+\frac{1}{m}\right)^2 & \left(K_d^2\left(\frac{1}{M}+\frac{1}{m}\right)^2 - k\left(\frac{1}{M}+\frac{1}{m}\right)\right) \\ -\frac{kK}{M}\left(\frac{1}{M}+\frac{1}{m}\right) + \frac{KK_d^2}{M}\left(\frac{1}{M}+\frac{1}{m}\right) & \sim & \sim & \sim \end{bmatrix}$$

Notice that the first column is identically zero if $K=0$. In this case $\begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^3 \end{bmatrix}$ is singular and the system is not observable.

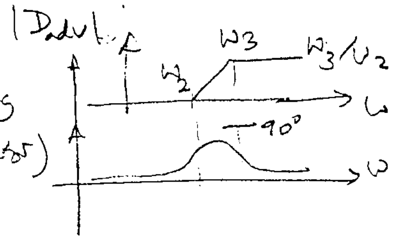
$K=0$ corresponds to "no spring attaching cart to support". In this case, replacing $z(t)$ by "$z(t)$ + constant" has no effect on the future transient behaviour of $y(t)$ (provided $\dot{z}(t)$ remains the same). So clearly we cannot determine $z(0)$ from

4 the data record $y(t)$, $0 \le t \le T$, i.e. system is not observable.

3.



(phase lag compensator)



phase advance compensator

5 Large $\frac{\omega_3}{\omega_2}$ gives very large control signals when there is a step change in the reference signal — this is often not acceptable.

$G(s) = \frac{2}{s(s+1)^2}$. We require $\angle G(j\omega_c)D_{adv}(j\omega_c) = \underbrace{-90^\circ - 2\tan^{-1}\omega_c + \angle D_{adv}(j\omega_c)}_{}$ $(= -180^\circ + 60^\circ)$

Hence $\angle D_{adv}(j\omega_c) = 2\tan^{-1}0.9 - 30^\circ = 53.974^\circ$ (when $\omega_c = 0.9$).

Since $\omega_c = \omega_{max}$, $\angle D_{adv}(j\omega_c) = 90^\circ - 2\tan^{-1}(\omega_2/\omega_3)$. It follows that $\omega_2/\omega_3 = 0.3251664$. Also $\omega_2\omega_2 = 0.9^2$.

$\Rightarrow \omega_2^2 = 0.3251664 \times 0.81$. Hence $\underline{\omega_2 = 0.51321}$. Also $\underline{\omega_3 = 1.5783}$

Also,

$$|G(j\omega_c)D_{adv}(j\omega_c)| = \frac{2}{0.9\times(1+0.81)}\cdot\frac{(1+(\omega_3/\omega_2)^2)^{1/2}}{(1+(\omega_2/\omega_3)^2)^{1/2}}$$

10 $= \frac{2}{0.9\times1.81}\,(1+3.075348^2)/(1+0.125166^2) = 3.7249715$

$G(s)D_{adv}(s)$ has the correct phase at $s = j\omega_c$. But its gain is 3.7249715

We reduce this gain to unity by means of phase lag compensation.

We require $\omega_0 < \omega_1 << \omega_c$ (to ensure $D_{lag}(j\omega_1) \simeq 0^\circ$) and $\omega_1/\omega_0 = 3.7249715$.

Choose

5 $\omega_1 = \frac{0.9}{20}$ (say) $= 4.5\times10^{-2}$ and $\omega_0 = \frac{\omega_1}{3.7249715} = 1.208$.

Final compensator design

$$D(s) = \underbrace{\frac{1+s/(4.5\times10^{-2})}{1+s/(1.208\times10^{-2})}}_{D_{lag}} \times \underbrace{\frac{1+s/0.51321}{1+s/1.5783}}_{D_{adv}}$$

4(a) The gain cross-over frequency is given by

$$\frac{k^2}{|j\omega_c + a|^2} = 1$$

i.e. $\underbrace{\omega_c = \sqrt{k^2 - a^2}}$ — (A)

The unit step, closed loop step response is

$$L.T.^{-1}\left\{ \frac{k}{s+a+k} \cdot \frac{1}{s} \right\} = \frac{k}{a+k}\left[1 - e^{-(a+k)t}\right]$$

Hence t_r is given by

$$1 - e^{-(a+k)t_r} = 0.99 \implies e^{(a+k)t_r} = 100$$

$\implies \underbrace{t_r = \log_e(100)/(a+k)}$ — (B)

Eliminate k from (A) and (B)

$$\omega_c^2 = k^2 - a^2, \quad k = \log_e(100)/t_r - a$$

$$\implies \underbrace{\omega_c^2 = \left(\frac{\log_e(100)}{t_r}\right)\left[\frac{\log_e(100)}{t_r} - 2a\right]}$$

For t_r small, $\log_e(100)/t_r >> a$ and $\frac{\log_e(100)}{t_r} - 2a \simeq \log_e(100)/t_r$

Hence

8 $\omega_c^2 \simeq \left(\frac{\log_e(100)}{t_r}\right)^2$, i.e. $\underline{\omega_c t_r} = \text{const.}$, with $\underline{\text{const.} = \log_e(10)}$

(b) Write $G(j\bar{\omega}) = X + jY$. We require

$$\tan (X+jY)/(1+X+jY) = \tan\phi$$

i.e. $\tan\left[\tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1+X}\right)\right] = \tan\phi$

Since $\tan(A-B) = (\tan A - \tan B)/1 + \tan A \tan B$, we have

$$\frac{\frac{Y}{X} - \frac{Y}{1+X}}{1 + Y^2/X(1+X)} = \frac{Y + XY - XY}{X^2 + Y^2 + X} = \tan\phi \; := N$$

i.e.

$$X^2 + Y^2 + X - \frac{1}{N}Y = 0.$$

This equation can be expressed

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2$$

12 (Circle, centre $\left(-\frac{1}{2}, +\frac{1}{2N}\right)$, radius $\underbrace{\frac{1}{2}\sqrt{1 + \frac{1}{N^2}}}$) can be expressed $1/\sin\phi$

5(a) $G(s) = (s^4 + 2s^3 - 3s^2)^{-1}$. If $x_1 = y$, $x_2 = \dot{y}$, $\dots$, $x_4 = \dddot{y}$

then $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3$, $\dot{x}_3 = x_4$, $\dot{x}_4 = -2x_3 + 3x_2 + u$. In

state space form:

$$\frac{d}{dt}\underline{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & +3 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \quad \equiv \quad \dot{x} = Ax + bu$$

4

Closed loop dynamics:

$$\dot{x} = (A - b[k_1 \vdots k_2 \vdots k_3 \vdots 0])x$$

We required $\det\left[sI - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & 3-k_3 & -2 \end{pmatrix} \right] = (s+\alpha(1+j))^2 (s+\alpha(1-j))^2$

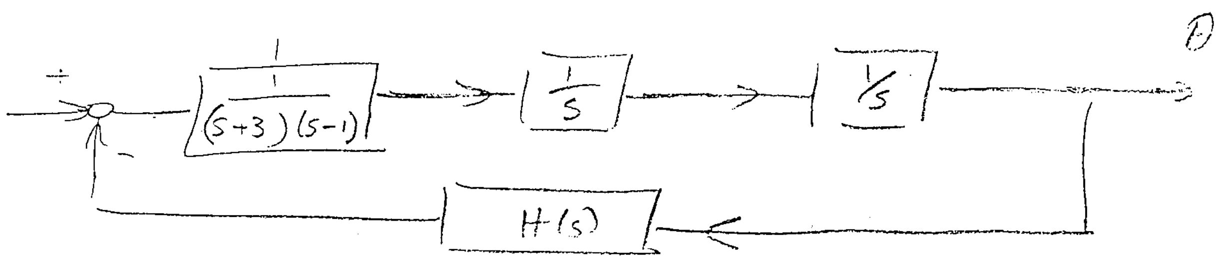
for some $\alpha \geq 0$, i.e.

$$s^4 + 2s^3 + (k_3 - 3)s^2 + k_2 s + k_1 =$$
$$s^4 + 4\alpha s^3 + 8\alpha^2 s^2 + 8\alpha^3 s + 4\alpha^4.$$

Matching gives

8 $\quad \alpha = \frac{1}{2}$, $k_1 = \frac{1}{4}$, $k_2 = 1$, $k_3 = 5$

(b) The block diagram can be re-arranged as



in which $H(s) = k(s+3) + k_v(s+3)s + k_a s^2$

Reorganising terms gives

$$H(s) = 3k + 3(k + k_v)s + (k_v + k_a)s^2$$

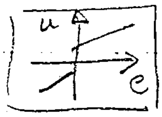
To locate CL poles as required, set

$$3k = \frac{1}{4}, \quad k + k_v = \frac{1}{3} \quad \text{and} \quad k_v + k_a = 5$$

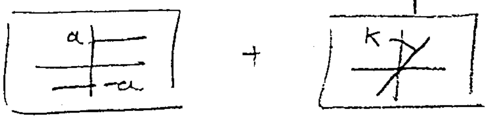
This gives

$$k = \frac{1}{12}, \quad k_v = \frac{1}{4}, \quad k_a = 4\tfrac{3}{4}$$

8

6. 

. This characteristic can be expressed as



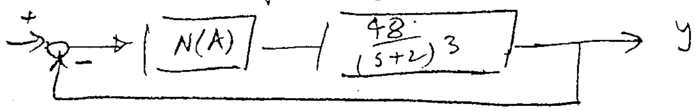
(a pure relay, with output amplitude a and a pure gain of K)

If $N_1(A)$ is the describing function of the relay

$$N_1(A) \cdot A = \frac{2}{T}\left(\int_0^{T/2} a \sin\omega t \, dt - \int_{T/2}^{T} a \sin\omega t \, dt\right) \qquad \left(T = \frac{2\pi}{\omega}\right)$$

$$= -\frac{4\omega}{2\pi} \times \frac{1}{\omega} a\cos\omega t \Big|_0^{T/2} = \frac{4a}{\pi}. \quad \text{So } N_1(A) = \frac{4a}{\pi A}.$$

6 / Adding the effect of the pure gain gives $N(A) = 4a/\pi A + K$



If $\bar{\omega}$ and A are the frequency and amplitude of limit cycle oscillations respectively, limit cycle analysis predicts

$$N(A) \times \frac{48}{(j\bar{\omega}+2)^3} = -1 + j0.$$

since $N(A)$ is real,

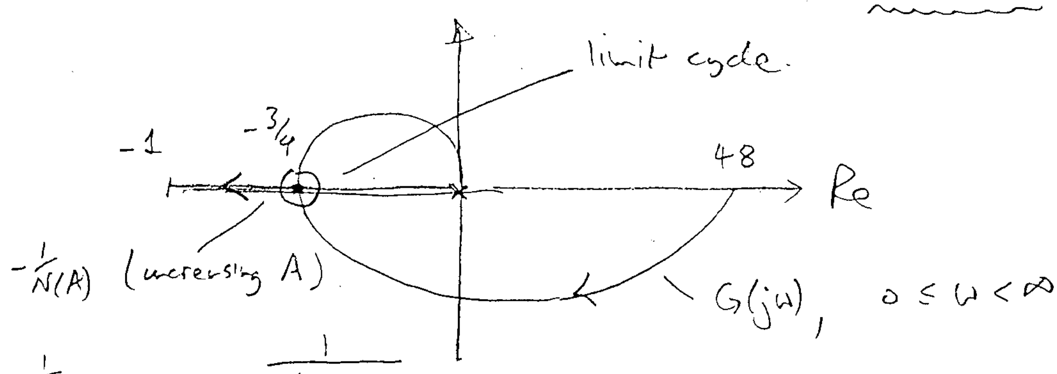
4 we deduce $-j\bar{\omega}^3 - 6\bar{\omega}^2 + 12j\bar{\omega} + 8 = 0$. Hence $\bar{\omega} = \sqrt{12}$. Also

$$-\left(\frac{4a}{\pi A} + K\right) \times \frac{48}{64} = -1$$

i.e. $4a/\pi A = \frac{4}{3} - K = \frac{1}{3}$,

It follows

4 $$a = \frac{\pi A}{12} = \pi/1200 = 2.62 \times 10^{-3}$$



$$-\frac{1}{N(A)} = -\frac{1}{1/3A + K}$$

We see that, as A increases, locus of $-1/N(A)$ passes from "unstable"

4 to stable region. It follows that the limit cycle is stable.