Paper Number(s): **E3.07**
ISE3.1

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2000

EEE/ISE PART III/IV: M.Eng., B.Eng. and ACGI

DIGITAL SIGNAL PROCESSING

Monday, May 15 2000, 10:00 am

There are SIX questions on this paper.

Answer FOUR questions.

All questions carry equal marks.

Corrected Copy
Q2 (a), Q2(c)

Time allowed: 3:00 hours

Examiners: Dr P.A. Naylor, Prof A.G. Constantinides

DIGITAL SIGNAL PROCESSING
2000

Special Instructions for Invigilators: None

Information for Candidates:

Sequence	**z-transform**
$\delta(n)$	1
$u(n)$	$\frac{1}{1-z^{-1}}$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$

Table 1 : z-transform pairs

$\delta(n)$ is defined to be the unit impulse function.
$u(n)$ is defined to be the unit step function.

Numbers in square brackets against the right margin of the following pages are a guide to the marking scheme.

1. Goertzel's algorithm is a recursive method for computing the DFT of a sequence. The algorithm can be derived by first considering the formula

$$X(k) = \sum_{l=0}^{N-1} x(l)\, W_N^{-k(N-l)} .$$

Show that this is equivalent to the standard formula for the DFT. [6]

By considering a new sequence

$$y_k(n) = \sum_{l=0}^{n} x_e(l) W_N^{-k(n-l)}$$

with

$$x_e(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$X(k) = y_k(n)\big|_{n=N}$$

write down the *z*-transform of $y_k(n)$ and the difference equation for $y_k(n)$ and hence draw the corresponding signal flow graph for $y_k(n)$. [7]

Deduce and briefly describe the operation of Goertzel's algorithm with reference to these formulae and signal flow graph. Include in your description a clear explanation of the procedure for computing $X(k)$ from $x(n)$. [7]

2. Consider a sequence $g(n)$ and its *z*-transform $G(z)$.

(a) Show that the z-transform of $n\, g(n) = -z\dfrac{dG(z)}{dz}$. [4]

(b) Derive $G(z)$ when $g(n) = r^n . \cos(\omega_0 n). u(n)$. [4]

(c) Consider two sequences $x(n)$ and $y(n)$ with *z*-transforms $X(z)$ and $Y(z)$. Let $w(n)$ be defined as the convolution of $x(n)$ with $y(n)$. The z -transform of $w(n)$ is $W(z)$.

Starting from the convolution sum, derive an expression for $W(z)$ and comment on the region of convergence of $W(z)$. [6]

Let $y(n) = a^n u(n)$ and let $x(n) = u(n)$. Find $W(z)$ and state its region of convergence. Sketch a pole/zero diagram of $W(z)$ and indicate the region of convergence on the diagram. You may assume *a* is real and $0 < a < 1$. [6]

3. Consider a discrete-time signal $x(n)$ of length N and its DFT $X(k)$.

(a) The DFT operation can be expressed in the following matrix form.

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where $\mathbf{X}$ and $\mathbf{x}$ are vectors and $\mathbf{D}_N$ is known as the DFT matrix. Write out in full the vectors $\mathbf{X}$ and $\mathbf{x}$ and the DFT matrix, $\mathbf{D}_N$, showing their elements in terms of $x(n)$, $X(k)$ and the term $W_N = e^{-j2\pi/N}$. [7]

(b) When $x(n)$ is complex it can be written

$$x(n) = g(n) + j\,h(n).$$

Show that

$$G(k) = \frac{1}{2}\left(X(k)_N + X^*(-k)_N\right) \qquad \text{and}$$
$$H(k) = \frac{1}{2j}\left(X(k)_N - X^*(-k)_N\right)$$

where $G(k)$ is the DFT of $g(n)$, $H(k)$ is the DFT of $h(n)$, X^* is the complex conjugate of X and the subscript N indicates modulo N indexing. [6]

(c) Now consider a real discrete-time signal $y(n)$. Using the formulae of (b), or otherwise, develop an efficient scheme for computing $Y(k)$, the DFT of $y(n)$. [7]

[Hint: consider $y(n) = y(2n) + y(2n+1)$ to be of length $2N$ and aim to compute $Y(k)$ from the sum of two N-point DFTs.]

4. (a) Compare FIR and IIR filters stating the advantages and disadvantages of each. Your answer should include definitions of both types of filters. [6]

(b) Consider an FIR discrete-time system with impulse response

$$\{h(n)\} = \{1, 1, 1, 1, 1, 1\}$$

Write down the difference equation and transfer function of this filter. [4]

Develop a recursive form of this difference equation and, hence, write down the transfer function of the recursive filter. Comment on the result. [6]

Draw a labelled sketch of the magnitude and phase of the frequency response of this filter. [4]

5. (a) In practical applications the derivative of a sequence is often approximated using sample difference equations.

Construct and write down such difference equations and their z-transforms for

(i) the first derivative, $y_1(n)$, of a signal $x(n)$, [6]

(ii) the second derivative, $y_2(n)$, of a signal $x(n)$.

For each of (i) and (ii) state whether the systems are linear, time-invariant and/or causal.

(b) One of the most common applications of median filtering is to smooth signals which have been corrupted by additive impulsive noise.

The output, $y(n)$, of a median filter is given by

$$y(n) = med\big(x(n-k), \ldots, x(n-1), x(n), x(n+1), \ldots, x(n+k)\big)$$

where the median function, *med*( ), lists the $2K+1$ samples in descending order and selects the value in the middle of the list, where K is an integer.

Comment on whether the above median filter is linear and/or time-invariant. Justify your comments analytically and construct a simple relevant example to illustrate your conclusions using the two sequences below. [7]

$$A = [-1, 7, 3, 0, -5] \qquad B = [-10, 2, -11, -12, 1]$$

(c) Consider the digital signal processing system shown in Figure 1. Determine the output, $y(n)$, in terms of the input, $x(n)$, and hence find the system function. [7]

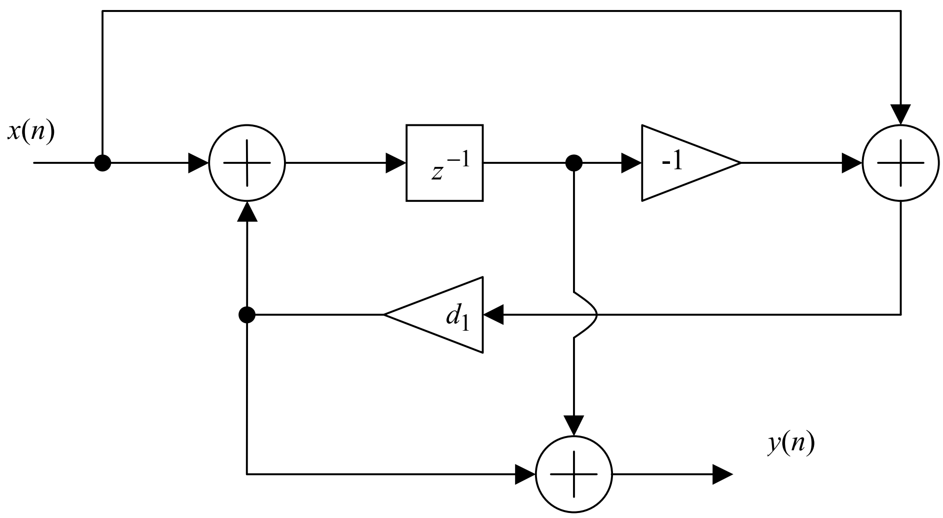


Figure 1

6. (a) Construct a signal flow diagram of a direct implementation of a multirate DSP system containing a two-band analysis filterbank followed by a synthesis filterbank. Label the diagram fully and write down expressions for suitable filterbank filters in terms of a half-band lowpass prototype filter $H_0(z)$. [5]

Show how polyphase filterbanks can be used to reduce computational complexity using appropriate diagrams and supporting analysis. [5]

(b) A signal $x(n)$ has a spectrum as shown in Figure 2. The signals, $p(n)$ and $q(n)$, are generated from $x(n)$ using the system of Figure 3. Draw labelled sketches of $\left|P(e^{j\omega})\right|$ and $\left|Q(e^{j\omega})\right|$ given that

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le |\omega| \le \pi \end{cases}.$$ [7]

Describe briefly the relationship between $\left|Q(e^{j\omega})\right|$ and $\left|X(e^{j\omega})\right|$. [3]

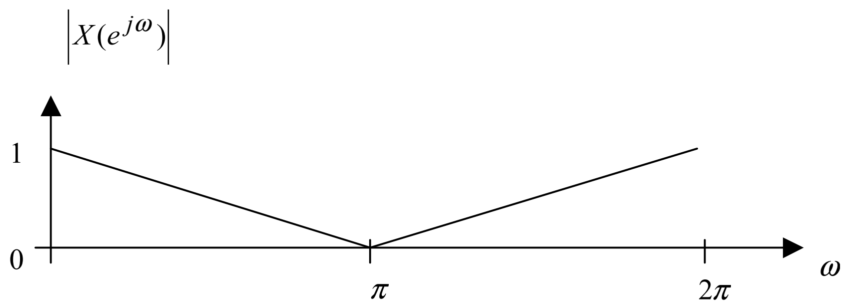


Figure 2

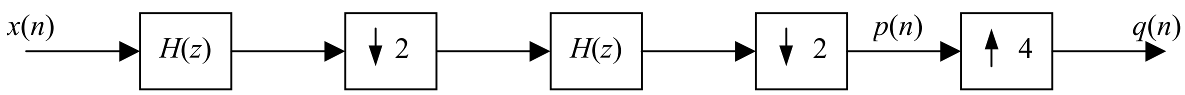


Figure 3

[END]

① $$X(k) = \sum_{l=0}^{N-1} x(l)\, W_N^{-k(N-l)}$$

$$= W_N^{-kN} \sum_{l=0}^{N-1} x(l)\, W_N^{kl}$$

1/14

$$= \sum_{l=0}^{N-1} x(l)\, e^{-j2\pi\frac{lk}{N}}$$

since $W_N = e^{-j2\pi/N}$ and

$$W_N^{-kN} = e^{-j2\pi k} = 1$$

$$y_k(n) = \sum_{l=0}^{n} x_e(l)\, W_N^{-k(n-l)}$$

Note that this is the convolution of the sequence $x_e(l)$ with a sequence $h_k(n)$ where

$$h_k(n) = \begin{cases} W_N^{-kn} & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

with $$H_k(z) = 1 + W_N^{-k} z^{-1} + W_N^{-2k} z^{-2} + \cdots$$

$$= \frac{1}{1 - W_N^{-k} z^{-1}}$$

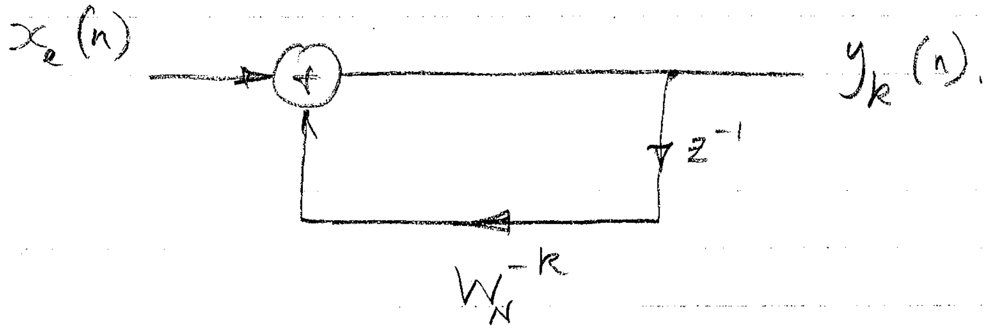
Hence $$Y_k(z) = X_e(z)\, H_k(z)$$

$$= \frac{X_e(z)}{1 - W_N^{-k} z^{-1}}$$

①

and

$$y_k(n) = x_e(n) + W_N^{-k}\, y_k(n-1)$$



$\frac{3}{14}$

② (a) $G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots$

$$\therefore \frac{dG(z)}{dz} = -g_1 z^{-2} - 2g_2 z^{-3} + \cdots$$

$$\therefore -z\frac{dG(z)}{dz} = g_1 z^{-1} + 2g_2 z^{-2} + \cdots = n\,g(n).$$

(b)
$$g(n) = r^n . \cos w_0 n . u(n)$$
$$= \frac{r^n}{2}\left(e^{jw_0 n} + e^{-jw_0 n}\right) u(n)$$
$$= \underbrace{\frac{1}{2} r^n e^{jw_0 n} u(n)}_{a(n)} + \underbrace{\frac{1}{2} r^n e^{-jw_0 n} u(n)}_{b(n)}$$

z-transform of $a(n)$:

$$A(z) = \mathcal{Z}\left\{\frac{1}{2}\alpha^n u(n)\right\} \text{ with } \alpha = r.e^{jw_0}.$$

$$= \frac{1}{2}\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}\frac{1}{1-re^{jw_0}z^{-1}}, |z|>r$$

Similarly $B(z) = \frac{1}{2}\frac{1}{1-re^{-jw_0}z^{-1}}$, $|z|>r$

$$\therefore Y(z) = A(z) + B(z) = \frac{1-(r\cos w_0)z^{-1}}{1-(2r\cos w_0)z^{-1}+r^2z^{-2}} \quad |z|>r$$

c)

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, z^{-n}$$

Let $m = n - k$:

$$W(z) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(n)\, z^{-m} \cdot z^{-k}$$

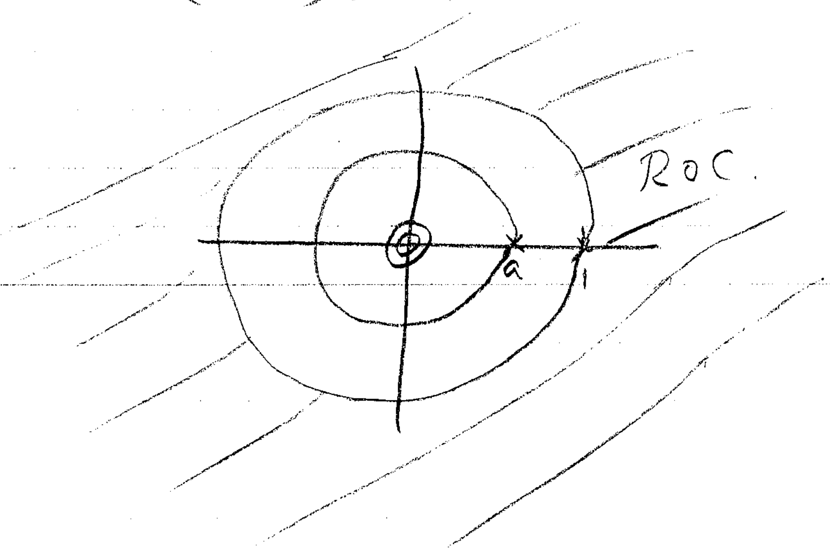
$$= X(z)\, Y(z)$$

ROC of $W(z)$ is intersection of ROC for $X(z)$ + $Y(z)$.

$$Y(z) = \frac{1}{1 - a z^{-1}} \qquad |z| > |a|$$

$$X(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$W(z) = \frac{z^2}{(z-a)(z-1)} \qquad |z| > 1.$$



3) a) $\bar{x} = [x(0) \;\; x(1) \;\; \cdots \;\; x(N-1)]^T$

$\bar{X} = [X(0) \;\; X(1) \;\; \cdots \;\; X(N-1)]^T$

5/14

$$\bar{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

b) $x(n) = g(n) + j\,h(n)$

$$X(k) = \sum_{n=0}^{N-1} (g(n) + j h(n)) e^{-j 2\pi kn/N}$$

$$X^*(k) = \sum_{n=0}^{N-1} (g(n) - j h(n)) e^{j 2\pi kn/N}$$

$$X^*(-k) = \sum_{n=0}^{N-1} (g(n) - j h(n)) e^{-j 2\pi kn/N}$$

$$\frac{1}{2}(X(k) + X^*(-k)) = \frac{1}{2}\Big(\sum_n g(n) W_N^{kn} + j \sum_n h(n) W_N^{kn} + \sum_n g(n) W_N^{kn} - j \sum_n h(n) W_N^{kn}\Big)$$

$$= \sum_n g(n) W_N^{kn}$$

$$= G(k)$$

Similarly for $H(k)$.

c) $y(n) = y(2n) + y(2n+1)$ — length $2N$.

$$Y(k) = \sum_{n=0}^{2N-1} y(n)\, W_{2N}^{kn} \quad \text{by definition.}$$

$$= \sum_{n=0}^{N-1} y(2n)\, W_{2N}^{2kn} + \sum_{n=0}^{N-1} y(2n+1)\, W_{2N}^{(2n+1)k}$$

Using $W_{2N}^{2kn} = e^{-j\,2\pi/2N \cdot 2kn} = e^{-j\,2\pi/N \cdot kn} = W_N^{kn}$ we have

$$Y(k) = \sum_{n=0}^{N-1} y(2n)\, W_N^{kn} + \sum_{n=0}^{N-1} y(2n+1)\, W_N^{kn} \cdot W_{2N}^{k}$$

2N-point DFT of real sequence

N-point DFT of even samples

N-point DFT of odd samples

Define two new sequences as

$g(n) = y(2n)$ with DFT $G(k)$
$h(n) = y(2n+1)$ with DFT $H(k)$

$G(k)$ and $H(k)$ can be found from the N-point DFT of $x(n) = g(n) + j\,h(n)$ using the formulae of b)

Finally $Y(k) = G(k) + W_{2N}^{k} H(k)$

for $k = 0, 1, \ldots, 2N-1$.

4) a) Bookwork. 7/14

b) $h(n) = [1, 1, 1, 1, 1, 1]$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$y(n) = x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4) + x(n-5)$$

For a recursive form, we write that

$$y(n-1) = x(n-1) + x(n-2) + \dots + x(n-6)$$

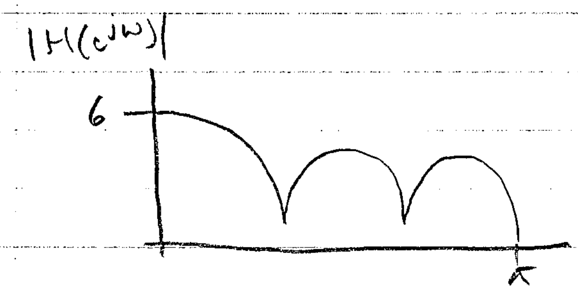
from which we see that

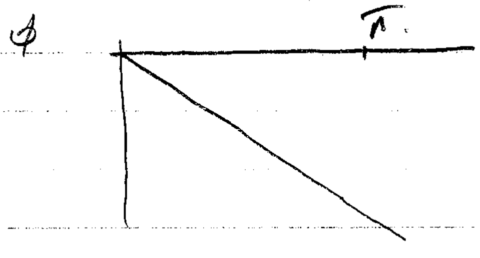
$$y(n) = x(n) + y(n-1) - x(n-6)$$

The equivalent IIR transfer function is therefore given by

$$Y(z) = X(z) + Y(z)z^{-1} - X(z)z^{-6}$$

$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-6}}{1 - z^{-1}}$$





$\frac{8}{14}$

(3) a)

(i) $y_1(n) = x(n) - x(n-1)$ $\qquad \frac{Y(z)}{X(z)} = 1 - z^{-1}$

LTI and causal

(ii) $y_2(n) = y_1(n) - y_1(n-1)$

$= x(n) - x(n-1) - x(n-1) + x(n-2)$

$= x(n) - 2x(n-1) + x(n-2)$

LTI and causal.

Alternatively:

$y_2(n) = x(n+1) - 2x(n) + x(n-1)$

LTI and non-causal.

b) Example :

$A = [-1 \quad 7 \quad 3 \quad 0 \quad -5] \qquad med(A) = 0$

$B = [-10 \quad 2 \quad -11 \quad -12 \quad 1] \qquad med(B) = -10$

$med(A+B) = med([-11 \quad 9 \quad -8 \quad -12 \quad -4]) = -8$

$med(A) + med(B) = -10$

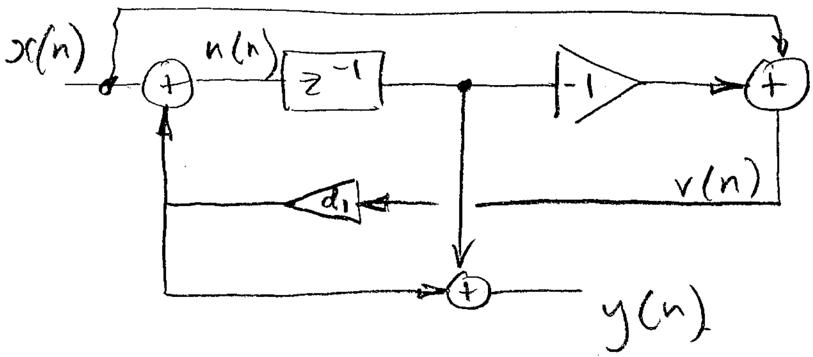
from which we see that the median filter does not obey the principle of superposition.

We could write $H(z) = \sum \nu(n) z^{-n}$

with $\mathcal{Z}\{\nu(n)\} = z^{-m(n)}$,

which is a filter, the impulse response of which is a single unit impulse at a delay of $m(n)$ samples. The delay $m(n)$ is determined by the median operation. This formulation illustrates a time-varying characteristic of median filtering.

c)



$$v(n) = x(n) - u(n-1) \qquad (1)$$
$$u(n) = x(n) + d_1 v(n) \qquad (2)$$
$$y(n) = d_1 v(n) + u(n-1) \qquad (3)$$
$$= d_1 x(n) + (1-d_1) u(n-1)$$

from (2) and (3) we have
$u(n) = x(n) + y(n) - u(n-1)$ and hence

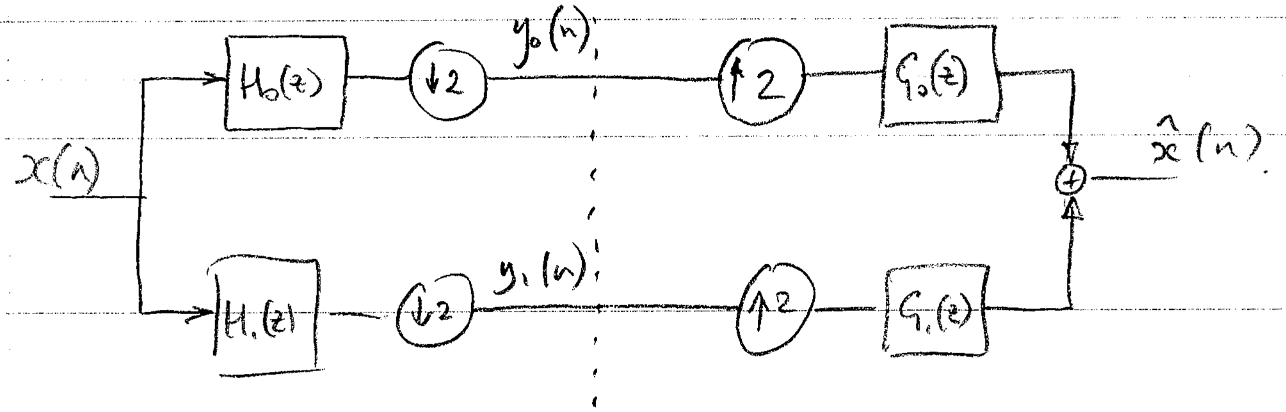
$$\frac{y(n+1) - d_1 x(n+1)}{1-d_1} = x(n) + y(n) - \frac{y(n) - d_1 x(n)}{1-d_1}$$

giving $y(n) = d_1 x(n) + (1-d_1)x(n-1) + (1-d_1)y(n-1) - y(n-1) + d_1 x(n-1)$

$$= d_1 x(n) + x(n-1) - d_1 y(n-1)$$

giving $$\frac{Y(z)}{X(z)} = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$$

6 a) $\frac{11}{14}$

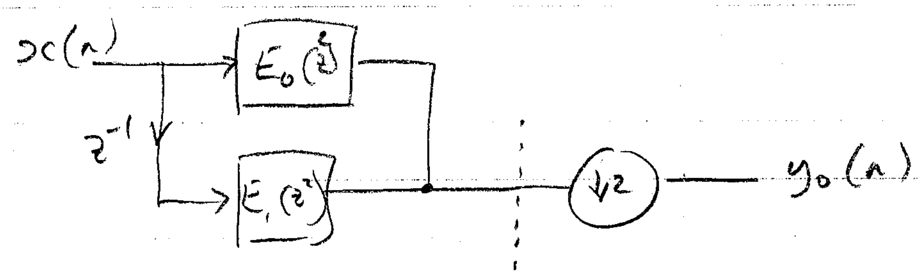


For QMF filters, choose: $H_1(z) = H_0(-z)$

For alias cancellation: $G_0(z) = H_1(-z)$
$G_1(z) = -H_0(-z)$.

For efficient polyphase implementation write

$$H_0(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

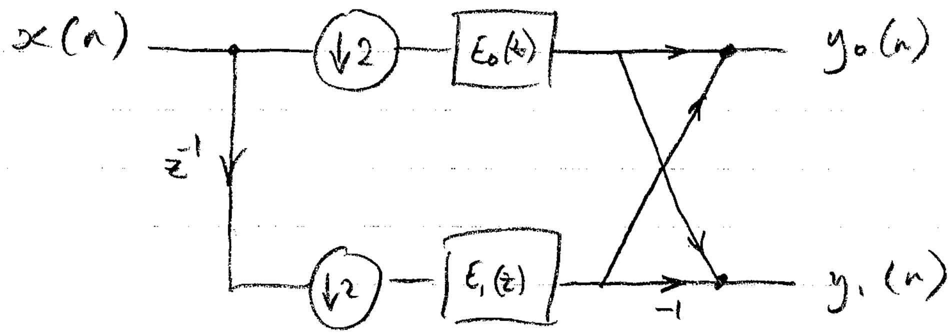


then employ the Noble Identities to change the order of the filtering and the decimation, thereby performing the filtering at the lower sampling rate.

Note that $H_0(z)$ and $H_1(z)$ can both be obtained from a single set of filters $E_0(z)$ and $E_1(z)$ by taking sum and difference respectively

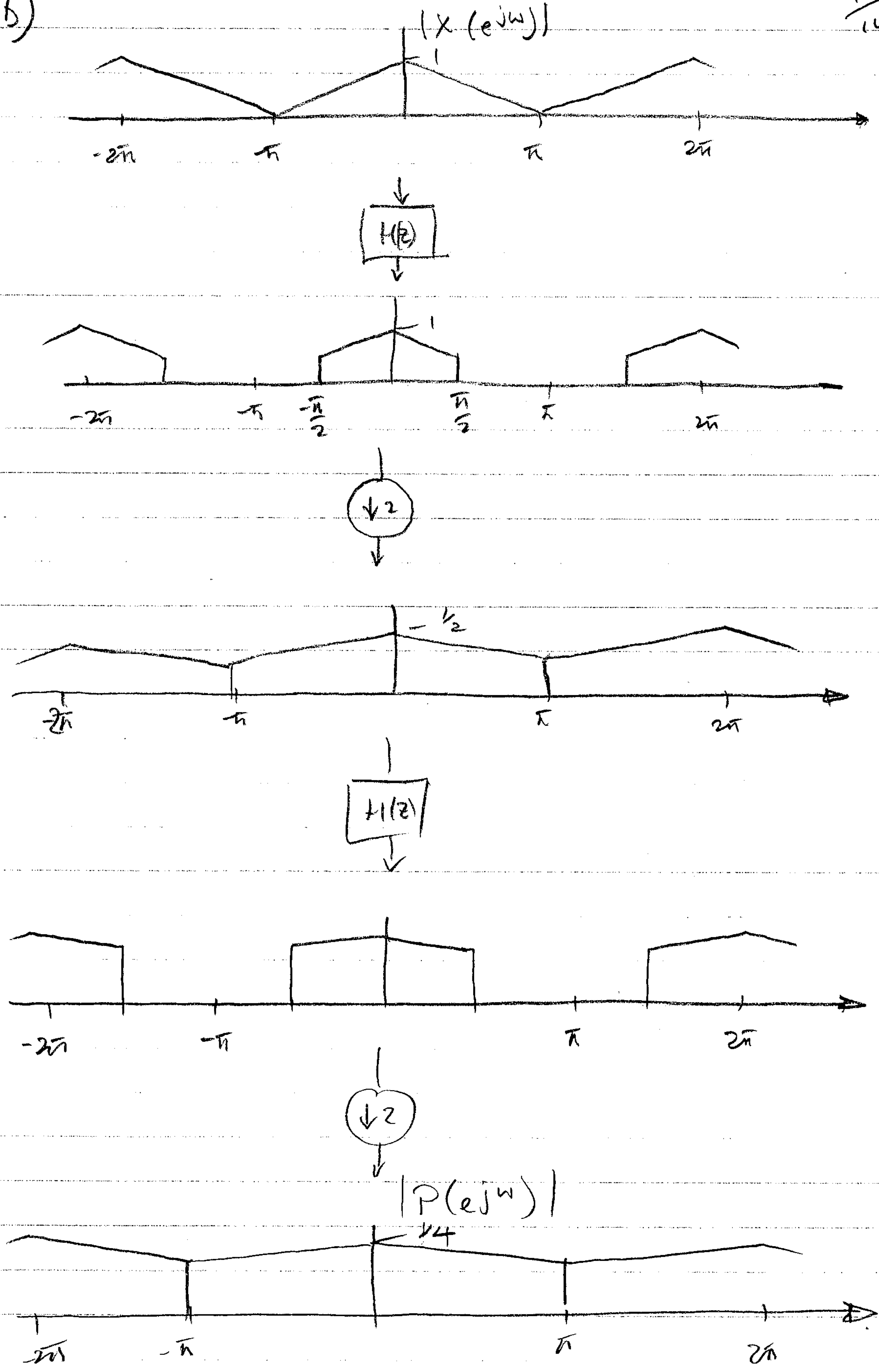
Hence:

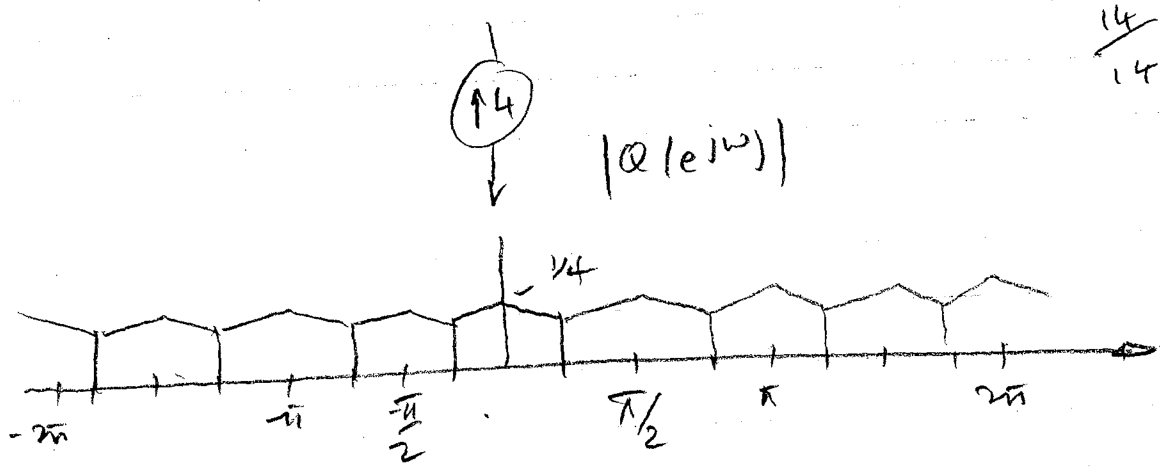
Analysis Bank



Similarly for the synthesis bank.

b) $\frac{13}{14}$





$|Q(e^{j\omega})|$ is identical to $|X(e^{j\omega})|$ over the range of frequencies from $2k\pi - \pi/4 < \omega < 2k\pi + \pi/4$
Outside this range we see periodic repetitions of the lower frequency band.