E2.6

IMPERIAL COLLEGE LONDON

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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART II: MEng, BEng and ACGI

**CONTROL ENGINEERING**

Friday, 25 May 2:00 pm

Time allowed: 2:00 hours

**There are FOUR questions on this paper.**

**Q1 is compulsory.**
**Answer Q1 and any two of questions 2-4.**
**Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).**

**Any special instructions for invigilators and information for candidates are on page 1.**

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| Examiners responsible | First Marker(s) : | I.M. Jaimoukha, I.M. Jaimoukha |
| | Second Marker(s) : | S. Evangelou, S. Evangelou |

1. (a) Consider the circuit in Figure 1.1 below. Assume that the opamps are ideal, $RC = 1$, $C2 = 0$ and that all initial voltages are zero.

   i. Derive the differential equation relating $V_i(t)$ and $V(t)$ and the differential equation relating $V(t)$ and $V_o(t)$. **[4]**

   ii. Derive the transfer function relating $V_i(s)$ to $V(s)$ and the transfer function relating $V(s)$ to $V_o(s)$. **[4]**

   iii. Derive the transfer function relating $V_i(s)$ to $V_o(s)$. **[2]**

   iv. Let $V_i(t) = 2e^{-2t}$ applied at $t=0$. Find $V_o(t)$. **[2]**

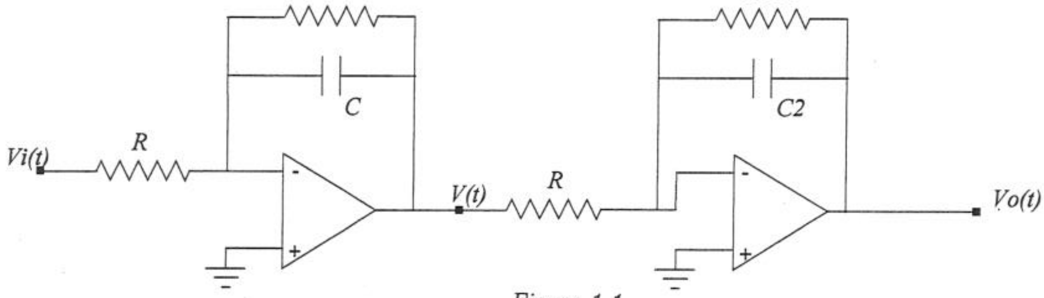


*Figure 1.1*

(b) In the feedback loop in Figure 1.2, $G(s) = \dfrac{1}{(s+1)^3}$ and $K$ is a variable gain.

   i. Sketch the locus of the closed-loop poles for $0 \leq K < \infty$. **[6]**

   ii. Derive the range of values of $K$ for which the closed-loop is stable. **[4]**

   iii. Derive the value of $K$ for which the closed–loop response is marginally stable. What is the frequency of oscillation? **[4]**

(c) Consider the feedback loop in Figure 1.2 with $G(s) = \dfrac{4}{(s+1)(s-2)}$.

   i. Sketch the Nyquist diagram of $G(s)$. **[8]**

   ii. Take $K = 1$. Use the Nyquist diagram to deduce the number of unstable closed–loop poles for the loop in Figure 1.2. **[6]**

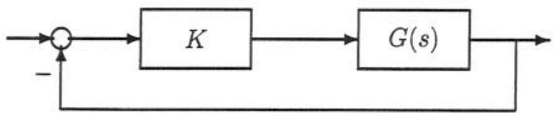


*Figure 1.2*

2. Consider the voltage feedback arrangement shown in Figure 2 for the speed control of a DC motor. The motor shaft drives a load with inertia $J$ and is connected to a tacho generator. Here, $v_r$ is the reference voltage, $v_a$, $i_a$ and $R_a$ are the armature voltage, current and resistance, respectively, $v_t$ is the tacho voltage, $w$ is the motor shaft speed and $E$ is the generated EMF. Also in the figure, $k > 0$ is a design parameter. Assume that

- The field flux is constant so that $E$ is proportional to $w$ and the developed torque, $T$, is proportional to $i_a$. Take the constant of proportionality to be the same and equal to $k_e$.
- The Power Op-Amp (POA) has negligible output resistance and dynamics, so that we can make the 'virtual earth' assumption.
- Torque disturbances and friction are negligible so that all the developed torque is supplied to the load.
- The tacho voltage is proportional to the speed with proportionality constant $k_t$.

(a) Derive the transfer function $G(s) = w(s)/v_a(s)$. **[5]**

(b) Derive an expression for $v_a(s)$ in terms of $v_r(s)$ and $w(s)$. **[5]**

(c) Hence, derive and clearly draw a block diagram representation of the feedback-loop. Take the reference signal to be $-v_r(s)$ and the output signal to be $w(s)$. Indicate clearly the signals $v_t(s)$ and $v_a(s)$. **[9]**

(d) Set $R_a = J = k_e = k_t = 1$. Derive the maximum value of $k$ such that the settling time of the closed-loop due to a step input is at most 1 second. **[11]**

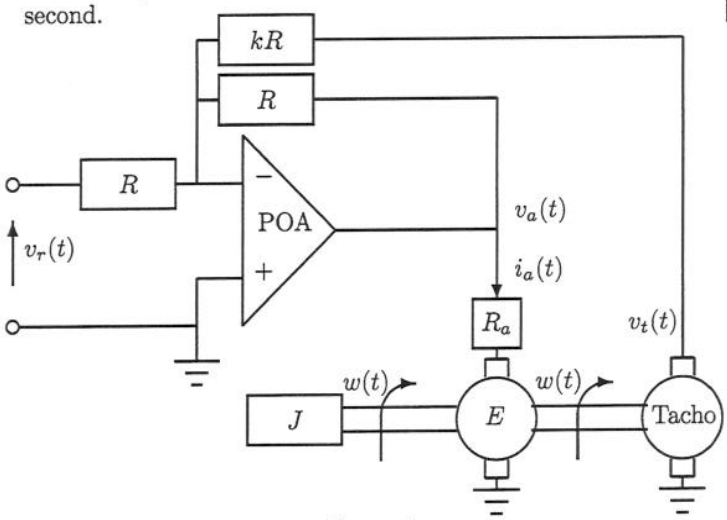


*Figure 2*

3. Consider the feedback control system shown in Figure 3 below. Here,

$$G(s) = \frac{1}{s(s+2)^2}$$

and $K(s)$ is the transfer function of the compensator.

(a) For $K(s) = k$, a constant compensator, draw the root locus accurately as $k$ varies in the range $0 \leq k \leq \infty$. **[6]**

(b) Take $K(s) = k$ where $k > 0$. Find the range of values of $k$ for which the closed loop is stable. **[6]**

(c) Take $K(s) = k$ where $k > 0$. Use the answer to Part (b) to find the value of $k$ for which the closed loop is marginally stable. For this value of $k$, what is the corresponding frequency of oscillation? **[6]**

(d) Design a proportional-plus-derivative compensator such that the following design specifications are simultaneously satisfied:

i. The closed loop is stable.

ii. The settling time for the dominant poles is at most $4s$.

iii. The damping ratio of the dominant poles is $\frac{1}{\sqrt{2}}$.

Draw the root locus of the compensated system. **[12]**

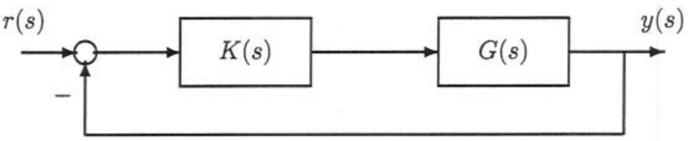


*Figure 3*

4. Consider the feedback control system in Figure 4.1 below. Here, $G(s) = 8/(s+2)^3$ and $K(s)$ is the transfer function of a compensator.

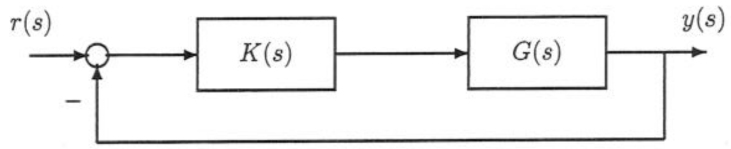


*Figure 4.1*

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts. [7]

(b) Set $K(s) = K$, a constant compensator. Give the number of unstable closed-loop poles for all (positive and negative) $K$. [7]

(c) Take $K = 1$. Determine the gain and phase margins. [8]

(d) Consider the bode plots shown in Figure 4.2 below for a first order compensator. Without doing any actual design, give a brief description of the compensator and its effects on the performance of the feedback loop and on the stability margins. You should also emphasize the difficulties involved in the design. [8]

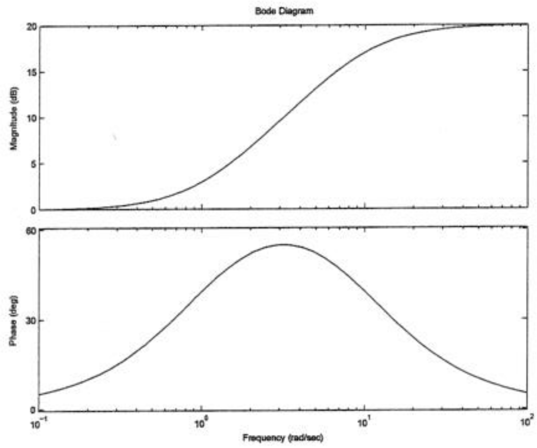


*Figure 4.2*

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**SOLUTIONS (E2.6, Control Engineering, 2007)**

1. (a) i. We make the virtual earth assumption and take $RC = 1$ and $C2 = 0$,

$$\frac{V_i(t)}{R} + \frac{V(t)}{R} + C\dot{V}(t) = 0, \qquad \frac{V(t)}{R} + \frac{V_o(t)}{.5R} = 0.$$

ii. Taking Laplace transforms in Part i,

$$\frac{V(s)}{V_i(s)} = -\frac{1}{s+1}, \qquad \frac{V_o(s)}{V(s)} = -\frac{1}{2}$$

iii. Multiplying the transfer functions in Part ii,

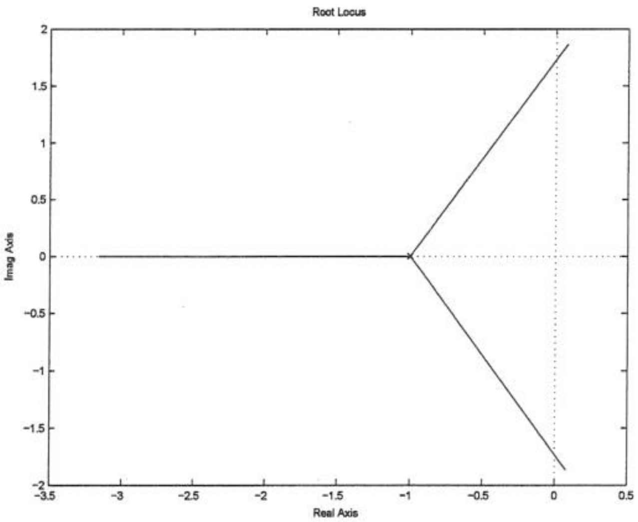
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{2(s+1)}$$

iv. Here, $V_i(s) = 2/(s+2)$. So, expanding in partial fractions,

$$V_o(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Taking inverse Laplace transforms, $V_o(t) = e^{-t} - e^{-2t}$.

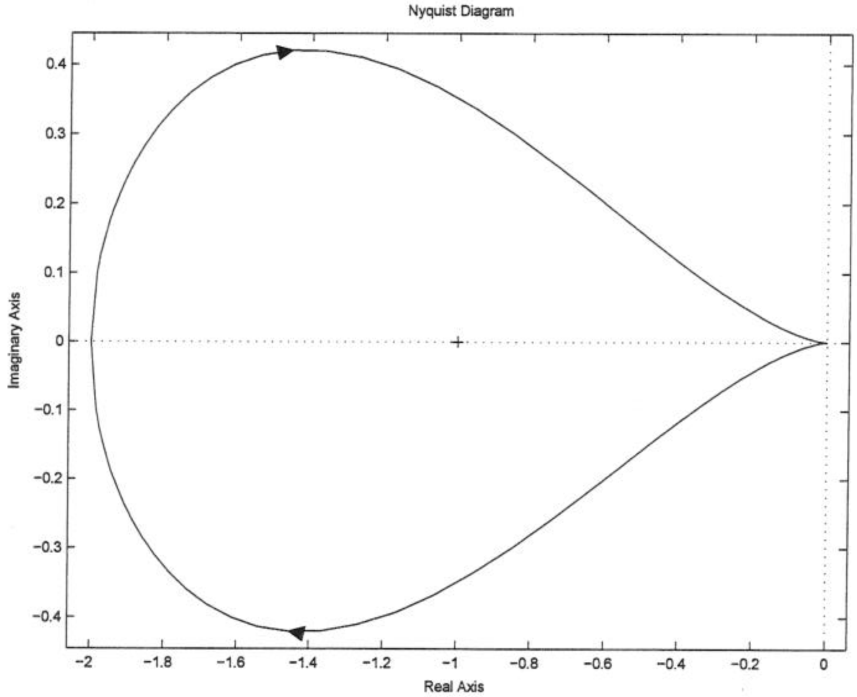
(b) i. The root locus is shown below.




ii. The Routh array gives the stability range as $-1 < K < 8$.

iii. We use the Routh array to find $K$ for marginal stability. So, $K = 8$. The auxiliary polynomial is $s^2 + 3$ and so the frequency of oscillation is $\sqrt{3}$.

(c) i. The Nyquist diagram is shown below:




ii. From the Nyquist theorem, $N = Z - P$ where $N(= 1$ in this case) is the number of clockwise encirclement of the point $-1$, $P = 1$ is the number of open loop unstable poles, and $Z$ is the number of closed–loop unstable poles. Hence $Z = 2$ and the closed–loop has two unstable poles.

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2. (a) The developed torque is $T(t) = k_e i_a(t)$ and the generated EMF is $E(t) = k_e w(t)$. Since friction is negligible and all the developed torque is supplied to the load, we have that $T(t) = J\dot{w}(t)$ or $k_e i_a(t) = J\dot{w}(t)$. However, $v_a(t) = R_a i_a(t) + E(t) = R_a i_a(t) + k_e w(t)$. It follows that $i_a(t) = \frac{1}{R_a} v_a(t) - \frac{k_e}{R_a} w(t)$.Thus

$$k_e \left( \frac{1}{R_a} v_a(t) - \frac{k_e}{R_a} w(t) \right) = J\dot{w}(t).$$

Rearranging and taking Laplace transforms,

$$J\dot{w}(t) + \frac{k_e^2}{R_a} w(t) = \frac{k_e}{R_a} v_a(t) \Rightarrow \left( Js + \frac{k_e^2}{R_a} \right) w(s) = \frac{k_e}{R_a} v_a(s)$$

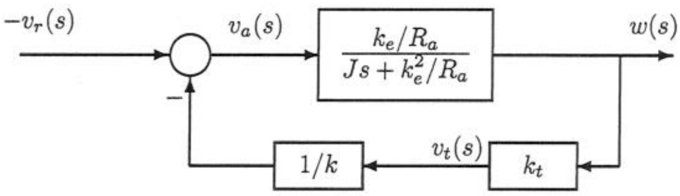
So

$$G(s) = \frac{k_e/R_a}{Js + k_e^2/R_a}.$$

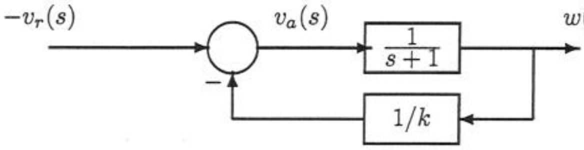
(b) Making the virtual earth assumption: $\frac{v_a(t)}{R} + \frac{k_t w(t)}{kR} + \frac{v_r(t)}{R} = 0$, since $v_t(t) = k_t w(t)$. Taking Laplace transforms and rearranging,

$$v_a(s) = -v_r(s) - \frac{k_t}{k} w(s).$$

(c) Using the last equation and the expression for $G(s)$, the block diagram becomes,




(d) Putting in the numbers, the block diagram simplifies to




The closed-loop pole is the root of the characteristic equation $s + 1 + k^{-1} = 0$ and is equal to $-(1 + k^{-1})$. Thus the settling time $T_s = 4/(1 + k^{-1})$. For $T_s \leq 1$ we need $k \leq 1/3$.

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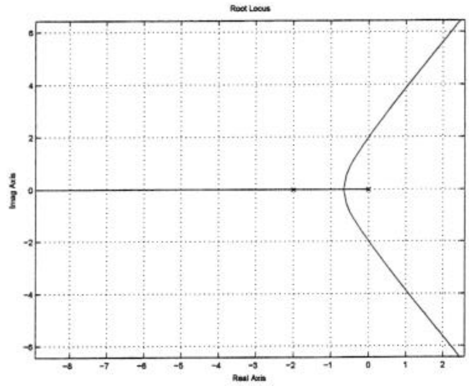
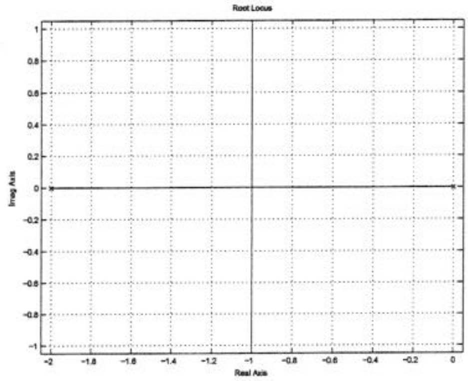
3. (a) The plot is shown below. The angles of the asymptotes are $\pm 60°$, $180°$ and the centre is at $-4/3$. The breakaway point is at $-2/3$.




(b) The characteristic equation is $s^3+4s^2+4s+k=0$. The Routh array:

| | | |
|---|---|---|
| $s^3$ | 1 | 4 |
| $s^2$ | 4 | $k$ |
| $s$ | $(16-k)/4$ | |
| 1 | $k$ | |

We require no sign changes in the first column. Thus $0 < k < 16$.

(c) From Part (b), when $k = 16$, the closed-loop is marginally stable. Putting $k=16$ in the Routh array, the auxiliary polynomial is $4s^2+16$ which has roots at $\pm j2$ and so the frequency of oscillations is 2 rad/s.

(d) A PD compensator has the form $K(s) = k(s + z)$. To satisfy the specifications the required closed–loop poles are at $-1 \pm j$. Next, we find $z$. Let the angle between $(-1 + j)$ and $z$ be $\theta$. Applying the angle criterion $\theta - (45° + 45° + 135°) = \pm 180°$ or $\theta = 45°$. Thus $z = 2$ and the compensated open loop is $1/(s(s + 2)$. The root locus is shown above. The gain criterion gives $k = -s(s + 2)|_{s=-1+j} = 2$.

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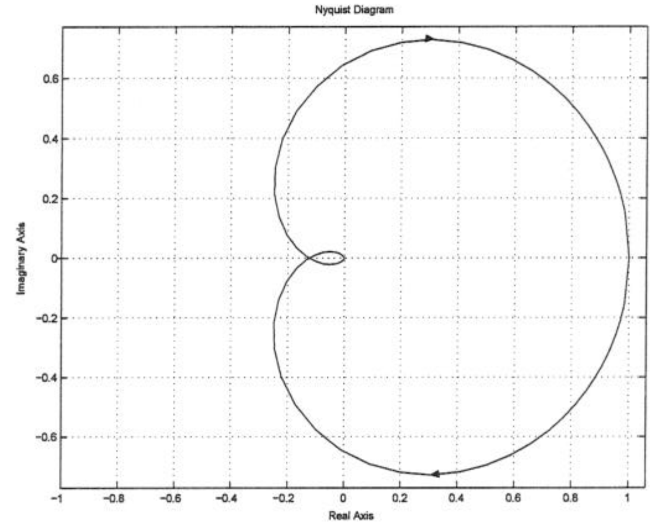
4. (a) The Nyquist plot is shown below. The real-axis intercepts are found by setting $Im[G(j\omega)] = 0$. Thus $\omega_i = 0, \pm 2\sqrt{3}, \infty$ so $G(j\omega_i) = 1, -0.125, -0.125, 0$.

(b) The number of unstable closed-loop poles associated with gain $K$ can be determined by the number of encirclements by $G(s)$ of the point $-1/K$. Thus $0 < K < 8 \Rightarrow$ stable, $K > 8 \Rightarrow 2$ unstable poles, $-1 < K < 0 \Rightarrow$ stable, $K < -1 \Rightarrow 1$ unstable pole.

(c) Since the negative real-axis intercept is at $-0.125$, then the gain margin is 8. For the phase margin we solve $|G(j\omega)| = 1$. However, the Nyquist diagram is inside the unit circle except when $\omega = 0$. Thus the phase margin is $180^o$.




(d) The bode plot is that of a phase-lead compensator $K(s) = \frac{1+s/\omega_0}{1+s/\omega_p}$ where $\omega_0 = 1$ and $\omega_p = 10$. It has gain close to unity for frequencies below $\omega_0$ and close to $\frac{\omega_p}{\omega_0} = 10$ beyond $\omega_p$. The phase is positive and large between $\omega_0$ and $\omega_p$ but small below and above. The increase in gain at frequencies above $\omega_p$ tends to degrade the stability margins as well as the noise attenuation properties, while the phase-lead tends to increase the phase margin, which is stabilising. It is thus important to balance the destabilising increase in gain against the stabilising increase in phase, which is a difficult task. We should place $w_p$ and $w_0$ in the crossover frequency range ($|G(j\omega)| \approx 1$.)