E2.6

IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

CONTROL ENGINEERING

Friday, 26 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : S. Evangelou,
Second Marker(s) : A. Astolfi,

1. (a) Consider the circuit in Figure 1.1 below. Assume that the opamps are ideal, $RC = 1$ and that all initial voltages are zero.

   i. Find the differential equation relating $V_i(t)$ and $V(t)$ and the differential equation relating $V(t)$ and $V_o(t)$. [4]

   ii. Find the transfer function relating $V_i(s)$ to $V(s)$ and the transfer function relating $V(s)$ to $V_o(s)$. [4]

   iii. Find the transfer function relating $V_i(s)$ to $V_o(s)$. [2]

   iv. Let $V_i(t)$ be a unit step applied at $t=0$. Find $V_o(t)$. [2]

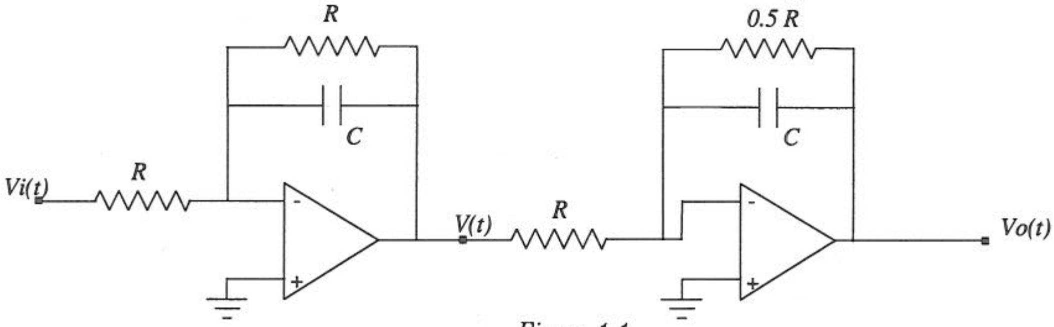

Figure 1.1

(b) In the feedback loop in Figure 1.2, $G(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$ and K is a variable gain.

   i. Sketch the locus of the closed–loop poles for $0 \leq K < \infty$. [6]

   ii. Find the value of K for which the response of the two dominant poles is critically damped. [4]

   iii. Find the value of K for which the closed–loop response is marginally stable. [4]

(c) Consider the feedback loop in Figure 1.2 with $G(s) = \dfrac{4}{(s-1)(s+2)}$.

   i. Sketch the Nyquist diagram of $G(s)$. [8]

   ii. Take $K = 1$. Use the Nyquist diagram to deduce the number of unstable closed–loop poles for the loop in Figure 1.2. [6]

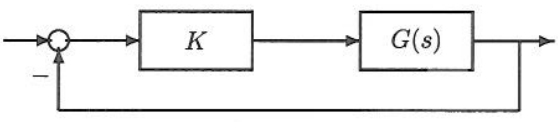

Figure 1.2

2. Consider the feedback system in Figure 2.1 for voltage regulation. Here, $v_r(t)$ is the reference voltage, $v_o(t)$ is the supplied output voltage and $i(t)$ is the load current. R_o is the output resistance of the op–amp. Figure 2.2 overleaf gives $E(t)$ in response to a unit step input in $v_e(t)$, where $E(t)$ is the opamp open–loop output voltage.

   (a) Use Figure 2.2 to derive a first order approximate transfer function relating $v_e(s)$ to $E(s)$. **[5]**

   (b) Derive an expression for $v_e(s)$ in terms of $v_r(s)$ and $v_o(s)$. **[5]**

   (c) Derive an expression for $v_o(s)$ in terms of $v_e(s)$ and $i(s)$. **[5]**

   (d) Hence, derive and draw a block diagram representation of the feedback loop. Take the reference to be $-v_r(s)$ and the output to be $v_o(s)$. Indicate the signals $v_e(s)$ and $i(s)$ on the block diagram. **[8]**

   (e) Express $v_o(s)$ in terms of $v_r(s)$ and $i(s)$. Take $R_o = 1\ \Omega$. Suppose $v_r(t)$ and $i(t)$ are both step inputs of sizes V_r and I, respectively. Find the steady state value of $v_o(t)$. Comment on the effect of the feedback amplifier. **[7]**

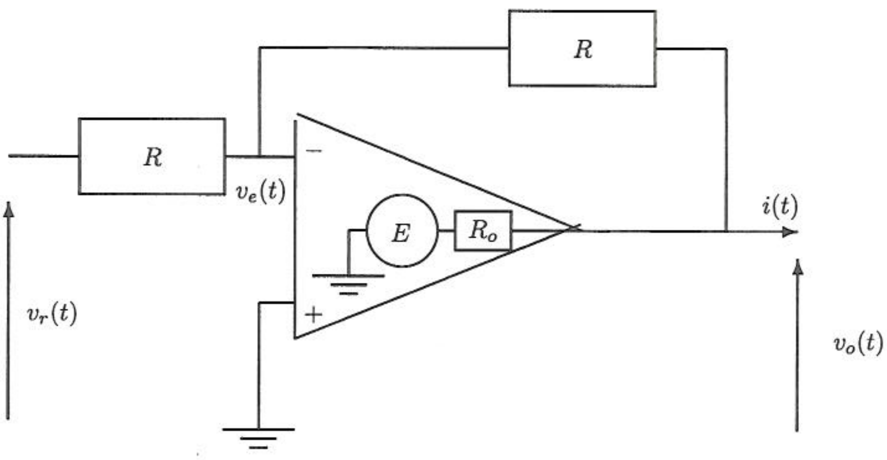

Figure 2.1

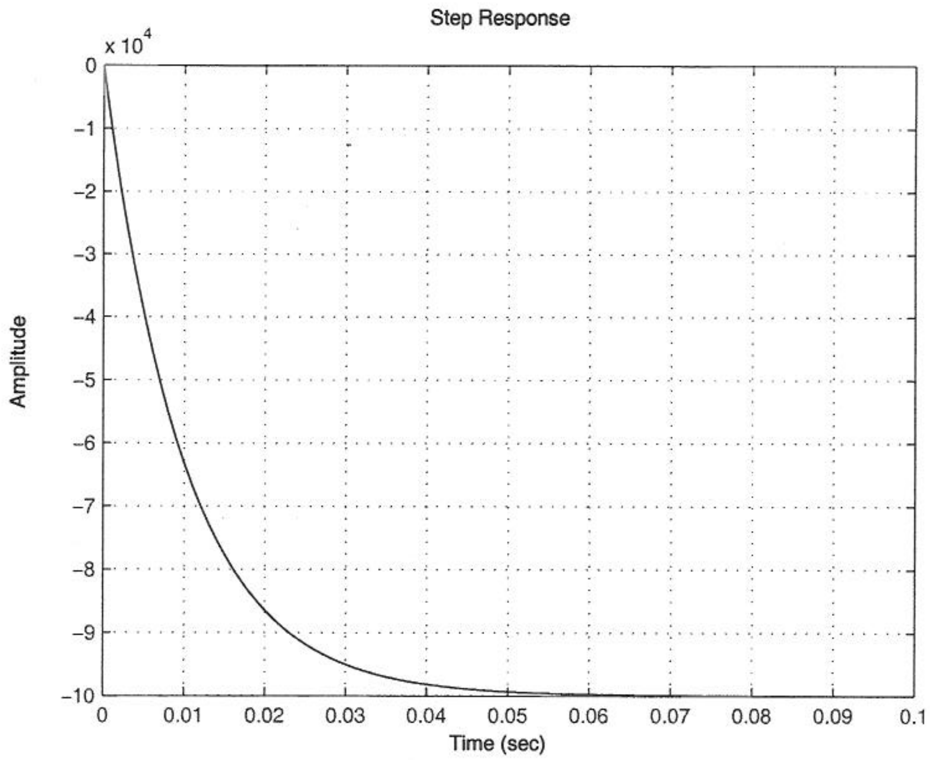

Figure 2.2 (for Question 2)

3. Consider the feedback control system shown in Figure 3 below. Here,

$$G(s) = \frac{1}{(s+1)(s+2)}$$

and $K(s)$ is the transfer function of the compensator.

(a) For $K(s) = k$, a constant compensator, draw the root locus accurately as k varies in the range $0 \le k \le \infty$. **[7]**

(b) Take $K(s) = k$ where $k > 0$. Find the largest value of k for which the closed loop response is non–oscillatory. **[7]**

(c) Design a first order compensator $K(s)$ such that the following design specifications are simultaneously satisfied:

i. The closed loop is stable.

ii. The settling time for the dominant poles is at most $4s$.

iii. The damping ratio of the dominant poles is $\frac{1}{\sqrt{2}}$.

Draw the root locus of the compensated system. **[16]**

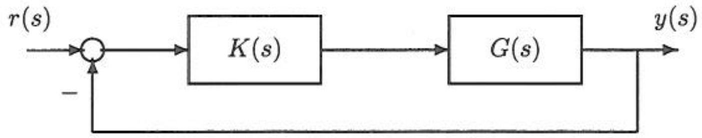

Figure 3

4. Consider the feedback control system in Figure 4 below. Here,

$$G(s) = \frac{1}{(s+1)^4}$$

and $K(s)$ is the transfer function of a compensator.

(a) Sketch the Nyquist diagram of $G(s)$, indicating the low and high frequency portions. Also, calculate the real–axis intercepts. [8]

(b) Take $K(s) = 1$ in Figure 4.

i. Show that the closed–loop is stable and determine the gain and phase margins. [5]

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ii. Comment ~~and~~ on the stability and performance properties of the closed–loop. Your answer should include analysis of tracking, disturbance rejection and noise attenuation. [6]

(c) i. Explain what is meant by a first order phase–lag compensator. Your answer should include an expression for such a compensator, and a description of its frequency response. [5]

ii. Without doing any actual design, describe how a phase–lag compensator would effect the stability margins and the steady–state tracking properties of the loop. [6]

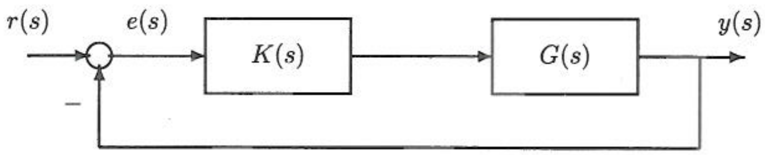


Figure 4

SOLUTIONS (E2.6, Control Engineering, 2006)

1. (a) i. We make the virtual earth assumption and take $RC = 1$,

$$\frac{V_i(t)}{R} + \frac{V(t)}{R} + C\dot{V}(t) = 0, \qquad \frac{V(t)}{R} + \frac{V_o(t)}{.5R} + C\dot{V}_o(t) = 0.$$

ii. Taking Laplace transforms in Part i,

$$\frac{V(s)}{V_i(s)} = -\frac{1}{s+1}, \qquad \frac{V_o(s)}{V(s)} = -\frac{1}{s+2}$$

iii. Multiplying the transfer functions in Part ii,

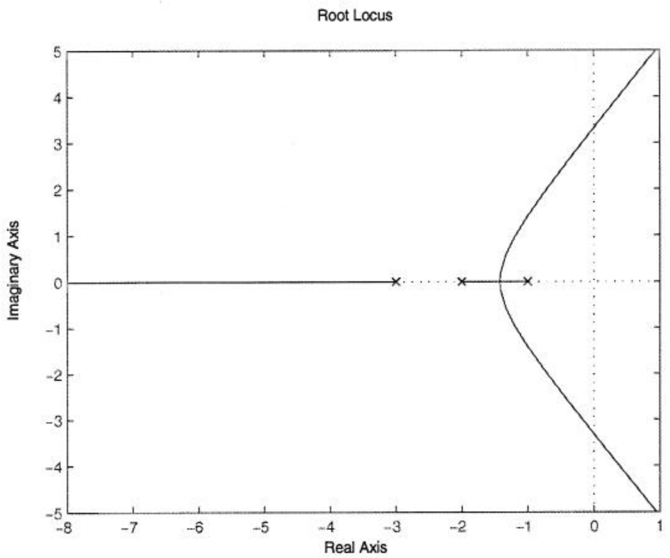
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{(s+1)(s+2)}$$

iv. Here, $V_i(s) = 1/s$. So, expanding in partial fractions,

$$V_o(s) = \frac{1}{s(s+1)(s+2)} = \frac{0.5}{s} + \frac{0.5}{s+2} - \frac{1}{s+1}$$

Taking inverse Laplace transforms, $V_o(t) = 0.5 + 0.5e^{-2t} - e^{-t}$.

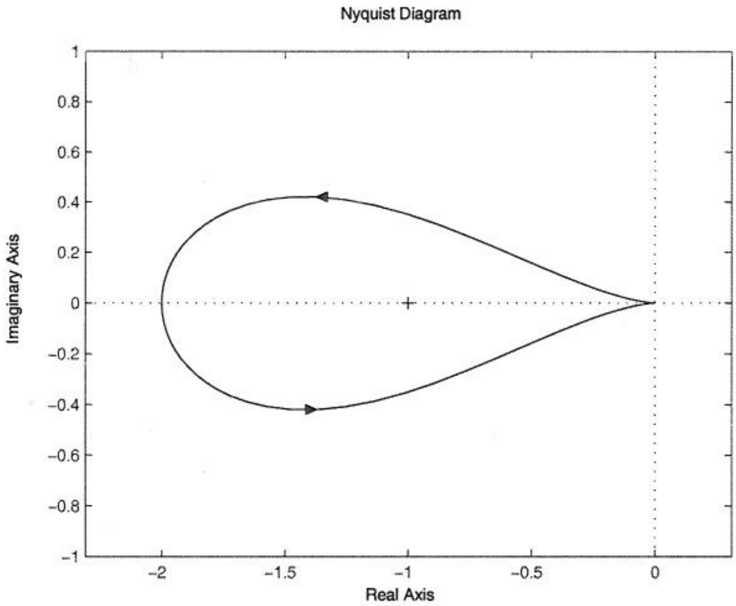
(b) i. The root locus is shown below.



ii. For critical damping, the poles are equal and real. We differentiate $G(s)$, set to zero and find real roots. So, $K \simeq 0.385$.

iii. We use the Routh array to find K for marginal stability. So, $K = 60$.

(c) i. The Nyquist diagram is shown below:



ii. From the Nyquist theorem, $N = Z - P$ where $N(= -1$ in this case) is the number of clockwise encirclement of the point -1, $P = 1$ is the number of open loop unstable poles, and Z is the number of closed–loop unstable poles. Hence $Z = 0$ and the closed–loop is stable.

2. (a) A first order transfer function has the form $\pm A/(1+\tau s)$. The value of A from the graph is 10^5, while τ is the time the response reaches 63% of its final value, so $\tau = 0.01$. Thus $E(s) = -\frac{A}{(\tau s+1)}v_e(s)$.

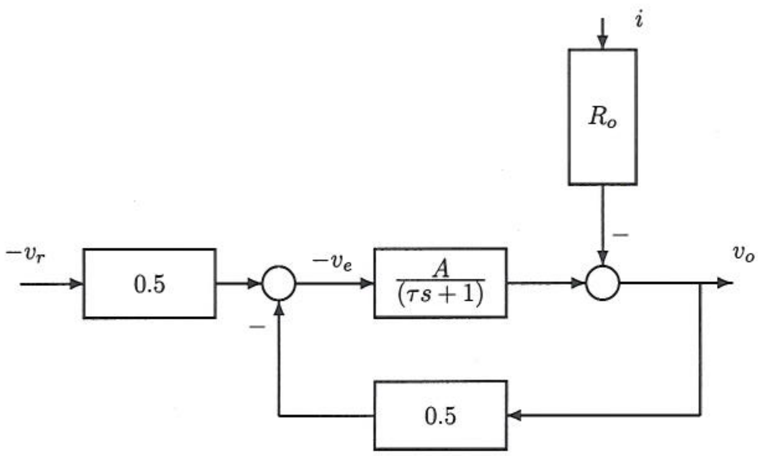
(b) Using a potential divider rule at the op–amp input gives

$$\frac{v_e(s) - v_r(s)}{v_o(s) - v_r(s)} = 0.5 \Rightarrow -v_e(s) = -0.5v_r(s) - 0.5v_o(s).$$

(c) At the op–amp output we have

$$E(s) - v_o(s) = R_o i(s) \Rightarrow v_o(s) = -\frac{A}{(\tau s+1)}v_e(s) - R_o i(s).$$

(d) Using parts (a) and (b), the block diagram becomes,



(e) Using the block diagram in part (d) and a manipulation gives

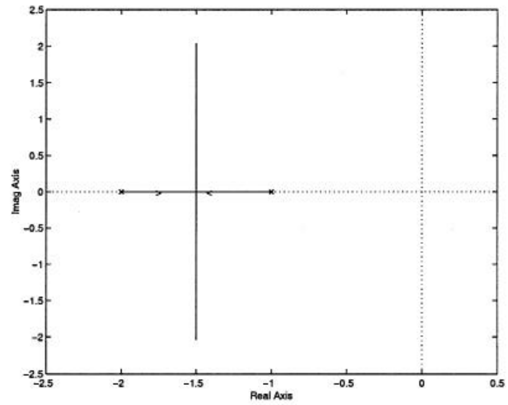
$$v_o(s) = -\frac{0.5A}{\tau s + 1 + 0.5A}v_r(s) - \frac{R_o(\tau s+1)}{\tau s + 1 + 0.5A}i(s).$$

Here, $i(s) = I/s$ and $v_r(s) = V_r/s$. We use the final value theorem to get

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} s v_o(s) = -\frac{0.5A}{1+0.5A}V_r - \frac{1}{1+0.5A}I \simeq -V_r$$

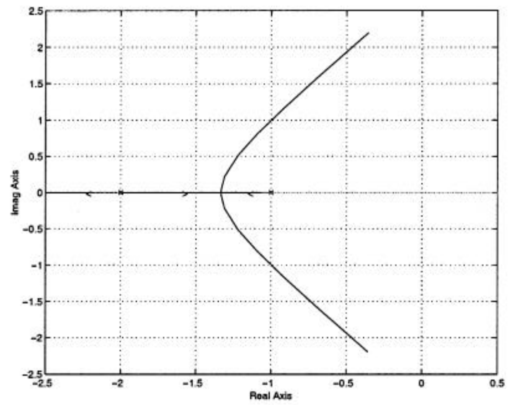
The feedback amplifier keeps the supplied voltage almost equal to $-V_r$ for a wide range of applied loads.

3. (a) The plot is shown below.

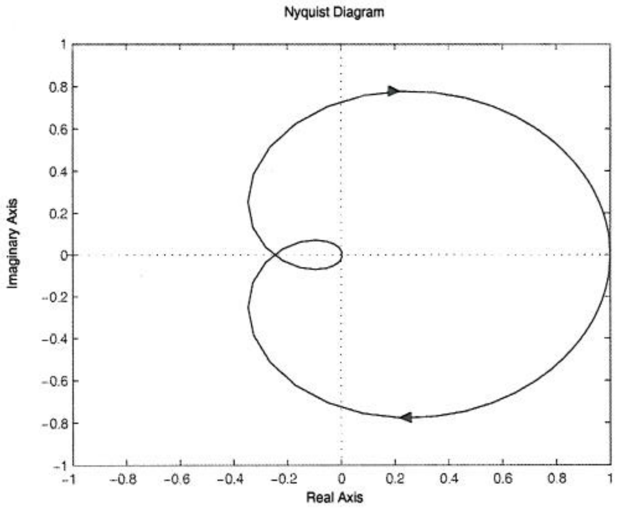


(b) For a non–oscillatory response, the closed–loop poles must be real. By inspection of the root–locus, the largest value of k is when the two poles are real and equal. The characteristic equation is given by $1+\frac{k}{s^2+3s+2}=0 \Rightarrow s^2+3s+2+k=0 \Rightarrow (s+1.5)^2+k-0.25=0 \Rightarrow k=0.25$.

(c) A first order compensator has the form $K(s) = k/(s-p)$. For a damping ratio of $1/\sqrt{2}$, the real and imaginary parts of the poles must have the same magnitude. The settling time is given by $4/r$ where r is the magnitude of the real part. Thus $r = 1$ for a settling time of $4s$. The required closed–loop poles are then at $-1 \pm j$. Next, we find p. Let the angle between $(-1+j)$ and p be θ. Applying the angle criterion $0-(90° + 45° + \theta) = \pm 180°$ or $\theta = 45°$. Thus $p = -2$. The root locus is shown below. We use the gain criterion to find k: $k = -(s+1)(s+2)^2|_{s=-1+j} = 2$.



4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by evaluating the Routh array and checking for marginal stability. This gives intercepts at frequencies $\omega_i = 0, \pm 1, \infty$ and so $G(j\omega_i) = 1, -0.25, -0.25, 0$.



(b) i. The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1, which is zero. Thus the closed–loop is stable since $G(s)$ has no unstable poles. Since the real–axis intercept is at -0.25, the gain margin is 4. For the phase margin, we need the intercept with the unit circle centred on the origin. In this case, only when $s = 0$ will this happen, so the phase margin is 180°.

ii. Since the stability margins are high, we expect good stability robustness against model uncertainties. However, since the gain is low at all frequencies, we expect poor tracking and disturbance rejection at low frequencies, and good sensor noise attenuation at high frequencies.

(c) i. A phase–lag compensator has the form

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_p < \omega_0$$

The phase–lag compensator has gain close to one for frequencies below ω_p and close to $\frac{\omega_p}{\omega_0} < 1$ for frequencies beyond ω_0. The phase is negative and large between these two frequencies but insignificant elsewhere.

ii. Thus phase–lag compensation can increase low frequency gain, and hence improve steady–state tracking since

$$|e(j\omega)| = |\frac{1}{1 + G(j\omega)K(j\omega)}|\ |r(j\omega)|$$

without increasing high frequency gain (and so degrading the gain margin). Care should be taken concerning the phase–margin since the phase lag may deteriorate this. We should therefore place w_p and w_0 in the 'middle' frequency range.