IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2005

EEE/ISE PART II: MEng, BEng and ACGI

Corrected Copy

CONTROL ENGINEERING

Friday, 27 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : J.C. Allwright, J.C. Allwright
Second Marker(s) : A. Astolfi, A. Astolfi

1. Consider the mass-spring-damper system shown in Figure 1 below, in which $y(t)$ denotes the displacement of the mass M from its rest position. A force $f(t)$ is applied to the mass M as shown. Take $M = 1Kg$ and $D_1 = D_2 = 1Ns/m$.

   (a) By considering the balance of forces on the mass, derive the differential equation relating $f(t)$ to $y(t)$. [6]

   (b) Derive the transfer function $G(s)$ between $f(s)$ and $y(s)$. [6]

   (c) Sketch the locus of the poles of $G(s)$ for $0 \leq K < \infty$. [8]

   (d) Find the value of K for which the response is (i) critically damped, (ii) marginally stable. [6]

   (e) Take $K = 1N/m$. Suppose that $f(t) = \cos t$. Find the steady-state response $y_{ss}(t)$. [7]

   (f) Take $K = 1N/m$. Sketch the Nyquist diagram of $G(s)$. [7]

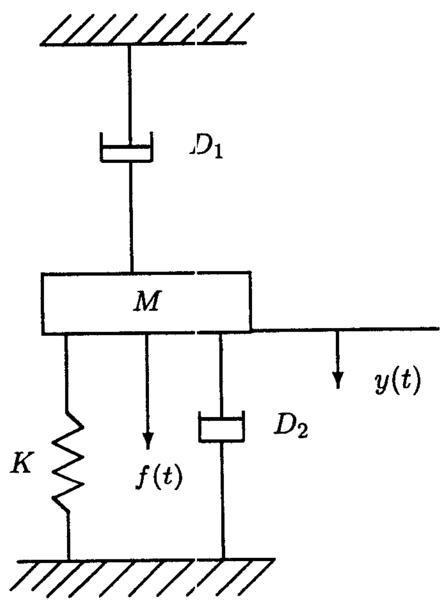


Figure 1

2. Consider the feedback system shown in Figure 2 below for the regulation of a voltage supply. Here, $v_r(t)$ is the reference voltage, $v_o(t)$ is the supplied output voltage and $i(t)$ is the load current. R_o is the output resistances of the op–amp. The op–amp voltage is modelled as

$$E(s) = -\frac{A}{(\tau s + 1)^3} v_e(s)$$

where $A > 0$ is the op–amp dc–gain, $\tau > 0$ is a time constant and $v_e(s)$ is the the Laplace transform of the voltage at the op–amp negative terminal.

(a) Derive an expression for $v_e(s)$ in terms of $v_r(s)$ and $v_o(s)$. **[6]**

(b) Derive an expression for $v_o(s)$ in terms of $v_e(s)$ and $i(s)$. **[6]**

(c) Hence, derive and draw a block diagram representation of the feedback loop. Take the reference to be $-v_r(s)$ and the output to be $v_o(s)$. Indicate the signals $v_e(s)$ and $i(s)$ on the block diagram. **[6]**

(d) Find the maximum value of the op–amp gain A for which the voltage regulator is stable. **[12]**

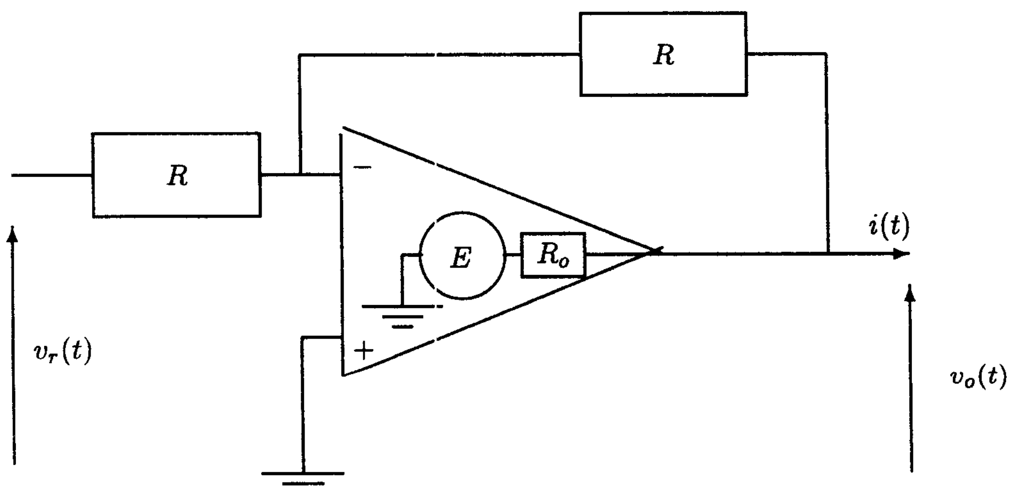


Figure 2

3. Consider the feedback loop shown in Figure 3 below. Here

$$G(s) = \frac{1}{s(s-1)}$$

and $K(s)$ is a compensator.

(a) Take $K(s) = k$ where k is a constant gain. Draw the root–locus accurately as k varies in the range $0 \leq k < \infty$. Your answer should include the centre and angles of the asymptotes, the breakaway points and the range of values of k for closed–loop stability. **[10]**

(b) Design a proportional–plus–derivative (PD) compensator $K(s)$ for the feedback loop shown in Figure 3 such that

i. the closed–loop is marginally stable, and

ii. the closed–loop response is oscillatory with a frequency of oscillation of 1 rad/s.

Sketch of the root–locus for the compensated system. **[10]**

(c) Design a PD compensator $K(s)$ for the feedback loop shown in Figure 3 such that

i. the closed–loop is stable, and

ii. the closed–loop has a double pole at -2.

Sketch of the root–locus for the compensated system. **[10]**

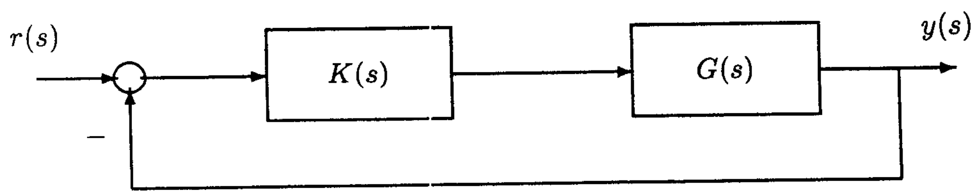


Figure 3

4. Consider the feedback control system in Figure 4 below. Here,

$$G(s) = \frac{4}{(s+1)^3}$$

and $K(s)$ is the transfer function of a compensator.

(a) Sketch the Nyquist diagram of $G(s)$, indicating the low and high frequency portions. Also, calculate the real–axis intercepts. [7]

(b) Take $K = 1$. Show that the closed–loop is stable and determine the gain and phase margins. [7]

(c) Without doing any actual design, briefly describe how a phase–lag compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_p < \omega_0$$

would effect the stability margins and the steady–state tracking properties of the loop. [8]

(d) Without doing any actual design, briefly describe how a phase–lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasize the difficulties involved in the design. [8]

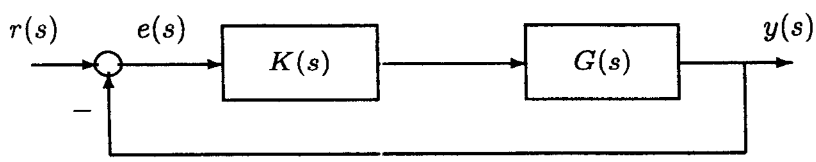


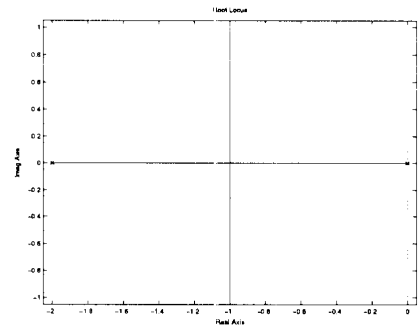
Figure 4

SOLUTIONS (E2.6/ISE2.9, Control Engineering, 2005)

1. (a) $\boxed{f(t) = M\ddot{y}(t) + (D_1 + D_2)\dot{y}(t) + Ky(t) = \ddot{y}(t) + 2\dot{y}(t) + Ky(t).}$

   (b) Taking Laplace transform: $(Ms^2+(D_1+D_2)s+K)y(s) = f(s)$. So $\boxed{G(s) = \frac{1}{Ms^2+(D_1+D_2)s+K} = \frac{1}{s^2+2s+K}.}$

   (c) The poles are the roots of $s^2 + 2s + K = 0$, or equivalently, of $1 + K\hat{G}(s) = 0$ where $\hat{G}(s) = 1/s(s+2)$. The locus of the poles of $G(s)$ is then the root locus of $\hat{G}(s)$ which is shown below.

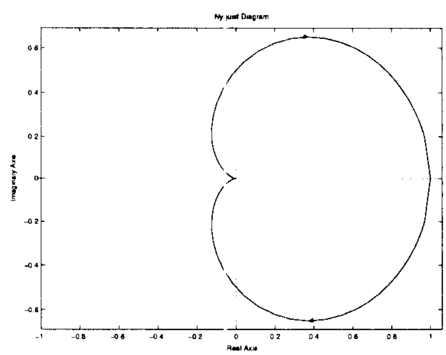


   (d) (i) $\boxed{K = 1.}$ (ii) $\boxed{K = 0.}$

   (e) Since $G(s) = 1/(s+1)^2$ is stable, the steady-state response to a sinusoid of frequency ω is also a sinusoid of the same frequency, with an amplitude $|g(j\omega)|$ and phase $\angle g(j\omega)$. Since $\omega = 1$,

$$\boxed{y_{ss}(t) = |g(j)|\cos(t + \angle g(j)) = 0.5\cos(t - \tfrac{\pi}{2}) = 0.5\sin t.}$$

   (f) The Nyquist diagram is shown below:



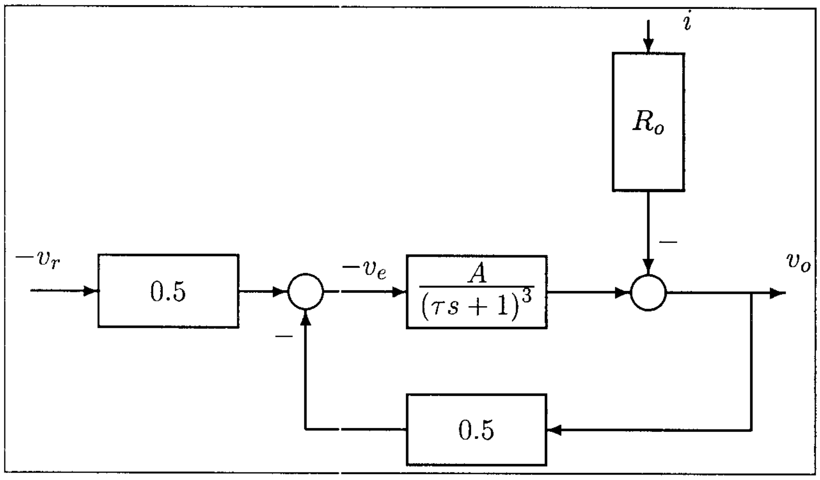
2. (a) Using a potential divider rule at the op–amp input gives

$$\frac{v_e(s) - v_r(s)}{v_o(s) - v_r(s)} = 0.5 \Rightarrow \boxed{-v_e(s) = -0.5v_r(s) - 0.5v_o(s).}$$

(b) At the op–amp output we have

$$E(s) - v_o(s) = R_o i(s) \Rightarrow \boxed{v_o(s) = -\frac{A}{(\tau s + 1)^3} v_e(s) - R_o i(s).}$$

(c) Using parts (a) and (b), the block diagram becomes,



(d) The closed–loop characteristic equation is

$$1 + \frac{0.5A}{(\tau s + 1)^3} = 0 \Rightarrow \tau^3 s^3 + 3\tau^2 s^2 + 3\tau s + 1 + 0.5A = 0.$$

The Routh array is then

s^3	τ^3	3τ
s^2	$3\tau^2$	$1 + 0.5A$
s	$\frac{8 - 0.5A}{3}\tau$	
1	$1 + 0.5A$	

For stability, we require no sign changes in the first column. Since $\tau > 0$ we require (1): $1 + 0.5A > 0$ and (2): $8 - 0.5A > 0$. Since A is positive, this reduces to $\boxed{A < 16.}$

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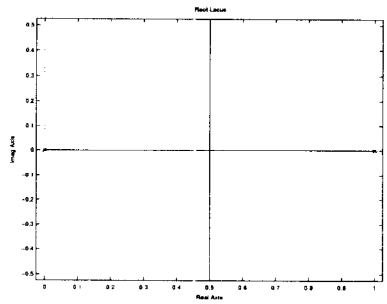
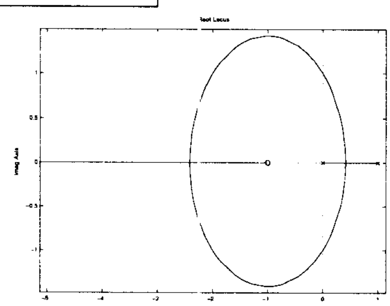
3. (a) The root-locus plot is shown below. The centre and angles of the

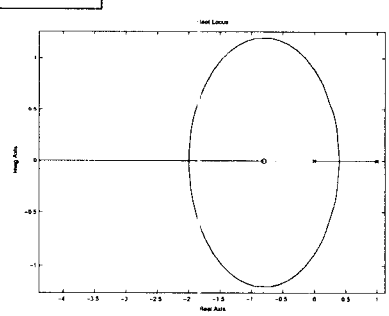


asymptotes are $\sigma = 0.5$ & $\psi = \pm 90^o$ and the breakaway point is $\sigma_b = 0.5.$ The closed–loop is unstable for all $k \geq 0.$

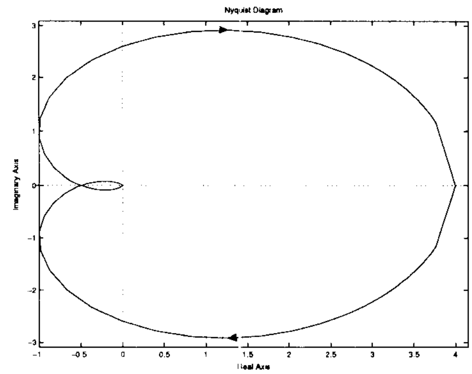
(b) A PD compensator has the form $K(s) = k(s + z)$ where $k > 0$ and $z > 0$. The required locations of the closed–loop poles are at $\pm j$. The angle criterion gives $z = 1$ and the gain criterion gives $k = 1$ so $K(s) = s + 1$. The root–locus is shown below.



(c) The required locations of the closed–loop poles are at -2, which is a break–in point. Setting $K(s) = k(s+z)$ and $dG(s)K(s)/ds = 0$ for $s = -2$ gives $z = 0.8$. The gain criterion gives $k = 5$. So $K(s) = 5(s + 0.8)$. The root–locus is shown below.



4. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ to zero. This gives intercepts at $\boxed{\omega_i = 0, \pm\sqrt{3}, \infty}$ and so $\boxed{G(j\omega_i) = 4, -0.5, -0.5, 0.}$



(b) The number of unstable closed-loop poles is determined by the number of encirclements by $G(s)$ of the point -1, which is zero. $\boxed{\text{Thus the closed-loop is stable}}$ since $G(s)$ has no unstable poles. Since the real-axis intercept is at -0.5, $\boxed{\text{the gain margin is } 2}$. For the phase margin, we need the intercept with the unit circle centred on the origin. We solve $|G(j\omega)| = 1$, this gives $\omega_1 = \sqrt{4^{\frac{2}{3}} - 1}$ and $\arg[G(j\omega_1)] = -153°$. $\boxed{\text{The phase margin is then } 27°.}$

(c) The phase–lag compensator has gain close to one for frequencies below ω_p and close to $\frac{\omega_p}{\omega_0} < 1$ for frequencies beyond ω_0. The phase is negative and large between these two frequencies but insignificant elsewhere. Thus phase–lag compensation can reduce high frequency gain (and so improve stability margins) without reducing low frequency gain (and hence degrading steady-state tracking since $|e(j\omega) = |\frac{G(j\omega)K(j\omega)}{1+G(j\omega)K(j\omega)}||r(j\omega)|)$ or introducing phase lag at high frequency (which destabilizes the loop). We should place w_p and w_0 in the 'middle' frequency range.

(d) The phase-lead has gain close to 1 for $\omega < \omega_0$ and close to $\frac{\omega_p}{\omega_0} > 1$ for $\omega > \omega_p$. The phase is positive and large between ω_0 and ω_p but small elsewhere. Thus the gain increase for $\omega > \omega_p$ degrades stability margins while the phase–lead increases the phase margin. It is important to balance the destabilizing increase in gain and the stabilizing increase in phase. We should place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).