IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

EEE/ISE PART II: MEng, BEng and ACGI

CONTROL ENGINEERING

Friday, 28 May 2:00 pm

Time allowed: 2:00 hours

Corrected Copy

There are FIVE questions on this paper.

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	J.C. Allwright
	Second Marker(s) :	A. Astolfi

1. The figure below illustrates an RLC circuit. The capacitor has capacitance C, the inductor has inductance L and the resistor resistance R. The input signal is the applied voltage $v_i(t)$ and the output signal is the voltage across the capacitor $v_o(t)$. The current through the circuit is indicated as $i(t)$. Take the charge across the capacitor to be $q(t)$.

(a) Derive the differential equation relating $q(t)$ to $v_i(t)$. **[3]**

(b) Derive the relationship between $q(t)$ and $v_o(t)$. **[1]**

(c) Derive a state-variable model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bv_i(t) \\ v_o(t) &= Dx(t)\end{aligned}$$

Indicate clearly your choice of the states. **[4]**

(d) Determine the transfer function relating v_i to v_o. **[4]**

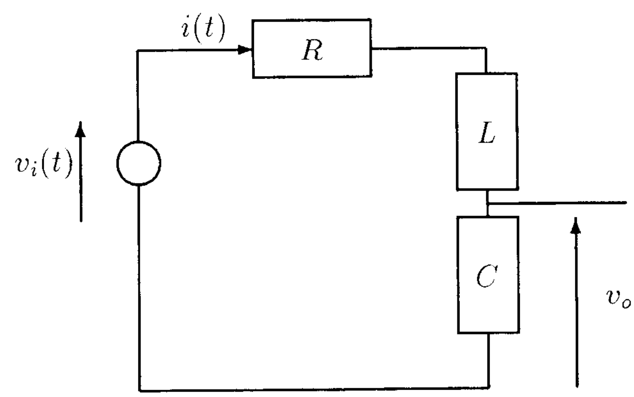
(e) Set $L = 0.5\ H$ and suppose that $v_i(t)$ is a unit step input voltage applied at $t = 0$. Derive the values of R and C so that the following design specifications are satisfied:

i. The capacitor voltage $v_o(t)$ settles to its steady–state value within 10^{-3} seconds.

ii. The maximum overshoot of $v_o(t)$ is 5% of its steady–state value.

What is the steady state value of $v_o(t)$?. **[8]**

(*Hint: You may take the settling time to be four times the time constant associated with the rate of decay of responses. Also, for a standard second order system, you may take the value of the damping ratio corresponding to a maximum overshoot of* 5% *to be* $\zeta = \frac{1}{\sqrt{2}}$.)



2. Consider the feedback system below for the regulation of a voltage supply. Here, $v_r(t)$ is the reference voltage, $v_o(t)$ is the supplied output voltage and $i(t)$ is the load current. R_1, R_2 and R_o are the input, feedback and output resistances of the op–amp, respectively. The op–amp voltage is modelled as

$$E(s) = -\frac{A}{\tau s + 1} v_e(s)$$

where A is the op–amp dc–gain, τ is a time constant and $v_e(s)$ is the the Laplace transform of the voltage at the op–amp negative terminal. Define

$$r_1 = \frac{R_1}{R_1 + R_2} \quad , \qquad r_2 = \frac{R_2}{R_1 + R_2}$$

In parts (a), (b) and (c) below, all references are to Laplace transforms of signals.

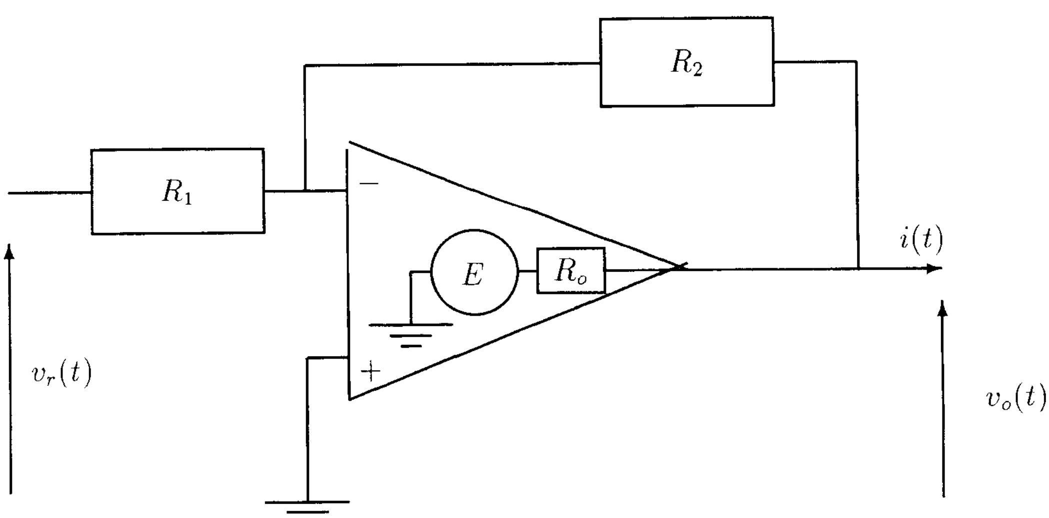
(a) Derive an expression for $v_e(s)$ in terms of $v_r(s)$ and $v_o(s)$. [3]

(b) Derive an expression for $v_o(s)$ in terms of $v_e(s)$ and $i(s)$. [3]

(c) Hence, derive and clearly draw a block diagram representation of the feedback loop. Take the reference signal to be $-v_r(s)$ and the output signal to be $v_o(s)$. Indicate clearly the signals $v_e(s)$ and $i(s)$ on the block diagram. [5]

(d) Derive an expression for $v_o(s)$ in terms of $v_r(s)$ and $i(s)$. [4]

(e) Set $R_o = 1\ \Omega$ and $A = 10^3$. Suppose that $v_r(t) = V_r$ is a constant reference voltage. Derive the minimum value of r_1 such that the steady–state output voltage drop between the no–load condition ($i(t) = 0$) and a full constant load current of 5 A is at most 1 V. (*Hint: You do not need the value of* τ). [5]



3. Consider the feedback loop in the figure below. Here

$$G(s) = \frac{1}{s^2 + s - 2}$$

and

$$K(s) = \frac{k}{s + a}$$

where k and a are design parameters.

(a) Derive the range of values of k and a for which the closed–loop is stable. Your answer should be of the form $k_1 < k < k_2$ and $a_1 < a < a_2$ where k_1, k_2, a_1 and a_2 are real numbers (or $\pm\infty$).

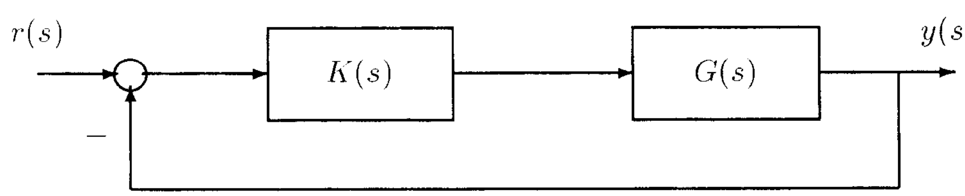
[6]

(b) Derive the range of values for k and a for which the closed–loop is marginally stable.

[6]

(c) By using the answer to part (b), or otherwise, find the values of k and a for which the response $y(t)$ to a unit impulse reference input $r(t)$ is oscillatory with the frequency of oscillation being 1 radians per second.

[8]



4. Consider the feedback loop in the figure below. Here

$$G(s) = \frac{s+3}{(s+1)(s+2)}$$

and $K(s)$ is a compensator.

(a) Take $K(s) = k$ where $k > 0$ is a constant gain.

(i) Draw the root–locus accurately as k varies in the range $0 < k < \infty$. Your plot should indicate clearly the direction of the locus, the location of any breakaway points and asymptotes.

[5]

(ii) Find the minimum value of k for which the closed–loop system step response is critically damped.

[3]

(b) Design a first order dynamic compensator $K(s)$ to achieve the following design specifications:

(i) The closed–loop system is stable.

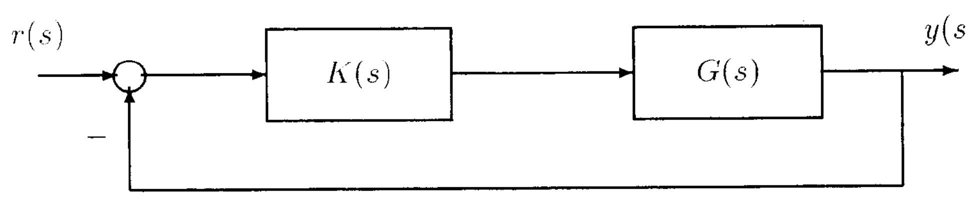
(ii) The steady–state error for a unit step reference signal is zero.

(iii) The closed–loop system step response is critically damped.

(iv) The steady–state error for a unit ramp reference signal is as small as possible.

(Hint: In order to simplify the calculations used in your design, you might wish to consider introducing a pole–zero cancellation. Once you have designed the compensator pole and zero, you might find it helpful to draw the resulting root–locus)

[12]



5. Consider the feedback control system in the figure below. Here,

$$G(s) = \frac{18}{(s+1)(s+2)^2}$$

and $K(s)$ is the transfer function of a feedforward compensator.

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts. **[5]**

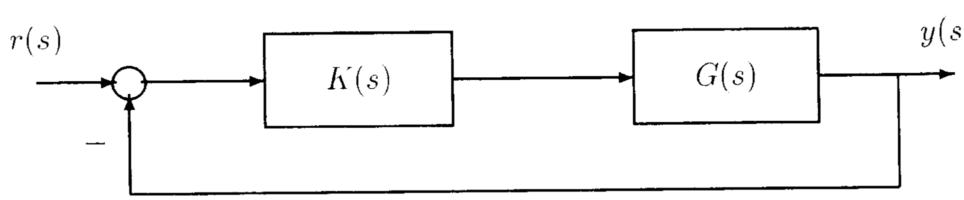
(b) Set $K(s) = k$, a constant compensator. Give the number of unstable closed-loop poles for all (positive and negative) k. **[5]**

(c) Take $K(s) = 1$. Determine the gain margin. **[5]**

(d) Without doing any actual design, briefly describe how a phase-lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should emphasise the difficulties involved in the design. **[5]**



SOLUTIONS (E2.6/~~ISE2.9~~, Control Engineering, 2004)

1. (a) Applying Kirchhoff's law on the loop,

$$\boxed{v_i(t) = L\ddot{q}(t) + R\dot{q}(t) + C^{-1}q(t).}$$

(b) The voltage across the capacitor is given by

$$\boxed{v_o(t) = C^{-1}q(t)}$$

(c) Take $x_1(t) = q(t)$ and $x_2(t) = \dot{q}(t)$. Then,

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_i(t) \\ v_o(t) &= \begin{bmatrix} \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}}$$

(d) Taking the Laplace transform of the equations relating v_i to q and v_o to q in parts (a) and (b) above,

$$\begin{aligned} (s^2L + sR + C^{-1})q(s) &= v_i(s) \\ C^{-1}q(s) &= v_o(s) \end{aligned}$$

The transfer function is obtained by dividing:

$$\boxed{G(s) = \frac{v_o(s)}{v_i(s)} = \frac{(LC)^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}}.}$$

(e) Comparing the transfer function $G(s)$ with the standard second order form

$$G(s) = \frac{(LC)^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

It follows that $\omega_n = \frac{1}{\sqrt{LC}}$ and $\zeta = 0.5R\sqrt{\frac{C}{L}}$. The first specification demands $\zeta = \frac{1}{\sqrt{2}}$ for 5% maximum overshoot while the second demands $\frac{4}{\zeta\omega_n} = 10^{-3}$. It follows that $\boxed{R = 4 \times 10^3\ \Omega}$ and $\boxed{C = 62.5 \times 10^{-9}\ F}$. The steady state output is simply $G(0)$ and so $\boxed{v_{o,ss} = 1.}$

2. (a) Using a potential divider rule at the op-amp input gives

$$\frac{v_e(s) - v_r(s)}{v_o(s) - v_r(s)} = \frac{R_1}{R_1 + R_2} = r_1$$

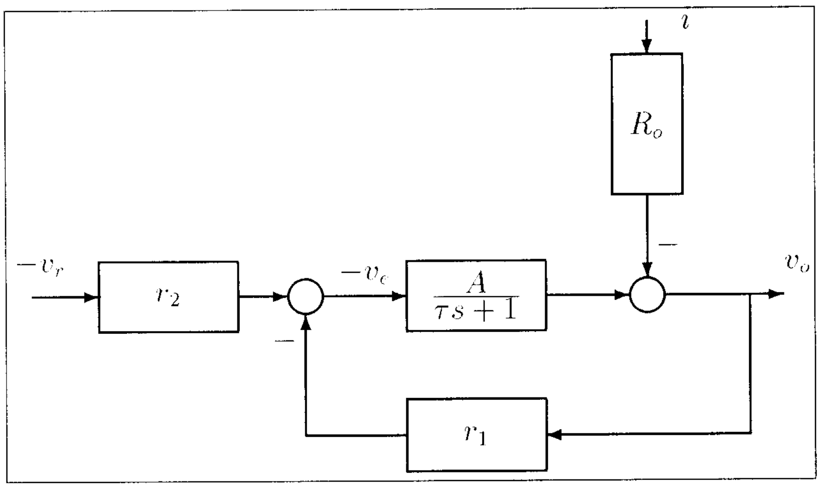
so that

$$\boxed{-v_e(s) = -r_2 v_r(s) - r_1 v_o(s).}$$

(b) At the op–amp output we have

$$E(s) - v_o(s) = R_o i(s) \Rightarrow \boxed{v_o(s) = -\frac{A}{\tau s + 1} v_e(s) - R_o i(s).}$$

(c) Using parts (a) and (b), the block diagram becomes,



(d) Using the block diagram in part (c) and a manipulation gives

$$\boxed{v_o(s) = -\frac{r_2 A}{\tau s + 1 + r_1 A} v_r(s) - \frac{R_o(\tau s + 1)}{\tau s + 1 + r_1 A} i(s).}$$

Since both $v_r(t) = V_r$ and $i(t) = 5$ are assumed to be constant, we have that $v_r(s) = V_r/s$ and $i(s) = 5/s$. Using the final value theorem, the steady–state no–load output voltage V_{NL} and the full–load output voltage V_{FL} are given by

$$\begin{aligned} V_{NL} &= -\frac{r_2 A}{1 + r_1 A} V_r \\ V_{FL} &= -\frac{r_2 A}{1 + r_1 A} V_r - \frac{1}{1 + r_1 A} 5 \end{aligned}$$

The voltage drop is obtained by subtracting to get

$$V_{NL} - V_{FL} = \frac{5}{1 + r_1 A} \leq 1$$

or $r_1 \geq 4A^{-1}$ and the minimum value of r_1 is $\boxed{r_1 = 4 \times 10^{-3}.}$

3. (a) The closed–loop characteristic equation is

$$1 + K(s)G(s) = 0 \Rightarrow s^3 + (a+1)s^2 + (a-2)s + (k-2a) = 0.$$

The Routh array is then

$$\begin{array}{c|cc} s^3 & 1 & a-2 \\ s^2 & a+1 & k-2a \\ s & \dfrac{a^2+a-2-k}{a+1} & \\ 1 & k-2a & \end{array}$$

For stability, we require no sign changes in the first column. Thus we require (1): $a+1>0$, (2): $a^2+a-2>k$ and (3): $k>2a$. Combining the second and third requirements gives $a^2+a-2>2a$ or $a^2-a-2>0$ or $(a+1)(a-2)>0$. Since $a+1>0$ from the first requirement, we have $\boxed{a>2}$ and so $\boxed{k>4}$ from the third requirement.

(b) The closed–loop system is marginally stable when all the elements of a row of the Routh array are equal to zero and all the elements in the first column of the modified array have the same signs.

(i) Taking the fourth row to be zero gives $k=2a$. The modified array becomes:

$$\begin{array}{c|cc} s^3 & 1 & a-2 \\ s^2 & a+1 & 0 \\ s & a-2 & \\ 1 & 0 \rightarrow a-2 & \end{array}$$

where the auxilliary polynomial is $p_3(s) = (a-2)s$ Thus $\boxed{a>2 \text{ and } k=2a}$ for marginal stability.

(ii) Taking the third row to be zero gives $k = a^2+a-2$. The modified array becomes:

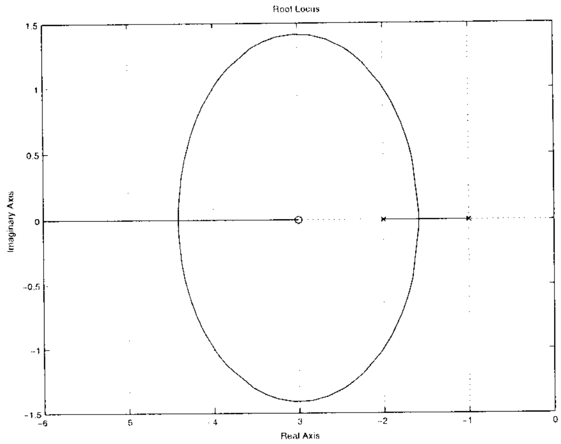
$$\begin{array}{c|cc} s^3 & 1 & a-2 \\ s^2 & a+1 & a^2-a-2=(a+1)(a-2) \\ s & 0 \rightarrow 2(a+1) & \\ 1 & (a+1)(a-2) & \end{array}$$

where the auxilliary polynomial is $p_2(s) = (a+1)(s^2+a-2)$ Thus $\boxed{a>2 \text{ and } k=a^2+a-2}$ for marginal stability.

(iii) Setting the second row to zero requires $a=-1$ and $k=-2$. The characteristic polynomial then is s^3-3s which has an unstable root and so the closed–loop in this case is not marginally stable.

(c) The closed–loop is oscillatory of it is marginally stable and if the auxilliary polynomial has pure imaginary roots. It follows from part (b)(ii) that the corresponds to the case $a>2$ and $k=a^2+a-2$ where the auxillary polynomial is $p_2(s) = (a+1)(s^2+a-2)$. For $p_2(s)$ to have a root at j (to give a frequency of oscillations of 1 radians per second), we need $a=3$. So $\boxed{a=3 \text{ and } k=3^2+3-2=10.}$

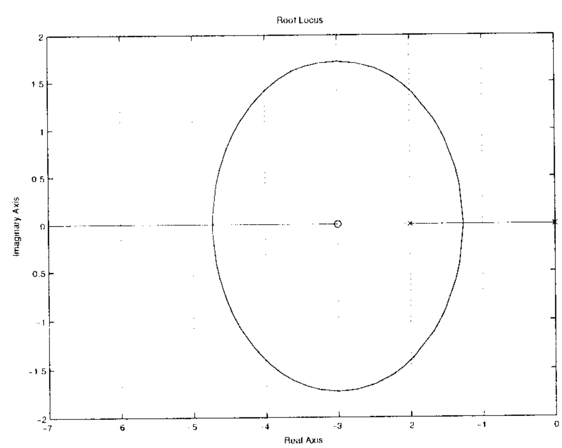
1. (a) (i) The root-locus plot is shown below. The breakaway points are the real



roots of $\frac{dG(s)}{ds} = 0$ and are equal to $\boxed{-3 \pm \sqrt{2} \sim -4.4, \quad -1.6}$. The $\boxed{\text{asymptote is at } 180^o.}$

(ii) For critical damping, the closed-loop poles are real and equal. This corresponds to the breakaway points. Since the smallest value of k is required, we need the right-most breakaway point. Using the gain criterion: $\boxed{k = -G(-3+\sqrt{2})^{-1} \sim 0.172.}$

(b) Let $K(s) = k(s-z)/(s-p)$. We design p, z and k. For zero steady-state error against a step reference signal, we need an integrator in the loop. So $\boxed{p = 0.}$ To simplify the calculation we use the hint and introduce a zero to cancel the pole at -1. So $\boxed{z = -1}$. The root-locus of the compensated system $G_c(s) = \frac{s+1}{s(s+2)}$ is shown below. The $\boxed{\text{breakaway points are } -3 \pm \sqrt{3} \sim -4.7, -1.3}$. For critical damping, we place the closed-loop poles at breakaway points. Since we are required to minimize steady-state error to a unit ramp reference, we need to $\boxed{\text{maximize the loop-gain}}$. So we need the left-most breakaway point. Using the gain criterion: $\boxed{k = -G_c(-3-\sqrt{3})^{-1} \sim 7.46.}$



E2.6 (4 of 5)

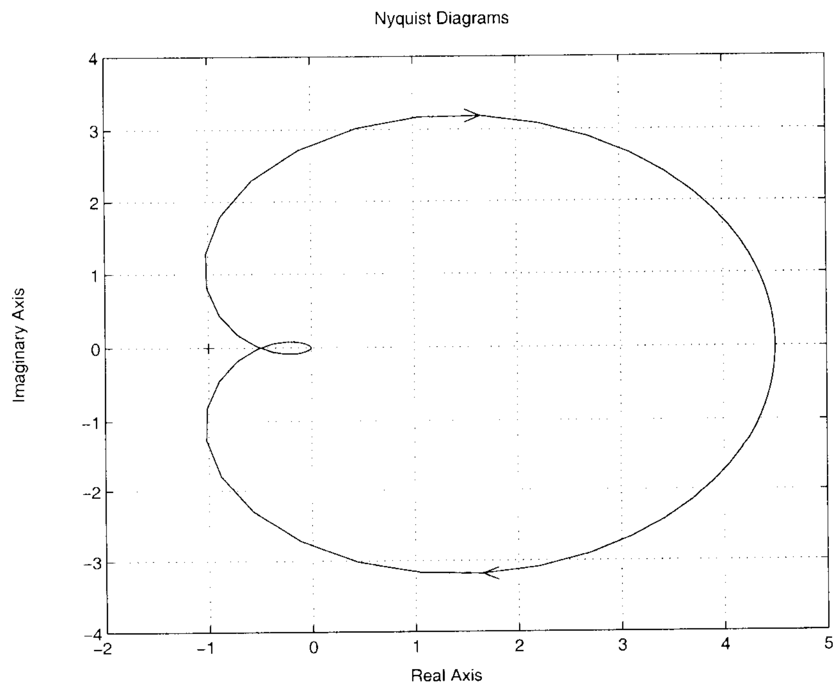
5. (a) The Nyquist plot is shown below. The real-axis intercepts can be found by setting the imaginary part of $G(j\omega)$ (or $1/G(j\omega)$) to zero. This gives intercepts for $\boxed{\omega_i = 0, \pm 2\sqrt{2}, \infty}$ and so $\boxed{G(j\omega_i) = 4.5, -0.5, 0.}$

(b) The number of unstable closed-loop poles associated with gain k can be determined by the number of encirclements by $G(s)$ of the point $-\frac{1}{k}$. Thus

$0 < k < 2$	$\Rightarrow$	no unstable poles
$k > 2$	$\Rightarrow$	2 unstable poles
$-\frac{2}{9} < k < 0$	$\Rightarrow$	no unstable poles
$k < -\frac{2}{9}$	$\Rightarrow$	1 unstable pole.

(c) Since the intercept with the negative real axis is at -0.5 the $\boxed{\text{gain margin is 2.}}$

(d) The phase-lead compensator has gain close to unity for frequencies below ω_0 and gain close to $\frac{\omega_p}{\omega_0} > 1$ for frequencies beyond ω_p. The phase is positive and large between these two frequencies but insignificant below and above. It follows that the increase in gain at frequencies above ω_p tends to degrade the gain and phase margins, while the phase-lead tends to increase the phase margin, which is stabilising. It is thus important to take care to balance the destabilising increase in gain against the stabilising increase in phase. We should therefore place w_p and w_0 in the crossover frequency range (when $|G(j\omega)| \approx 1$).



E2.6 (5 of 5)