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Paper Number(s): **E2.6**
ISE2.9

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART II: M.Eng., B.Eng. and ACGI

CONTROL ENGINEERING

Friday, 24 May 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

Time allowed: 2:00 hours

Examiners responsible:

First Marker(s): Jaimoukha,I.M.

Second Marker(s): Allwright,J.C.

1. Consider the feedback control system shown in the figure below. Here, $r(s)$, $y(s)$ and $e(s)$ represent the Laplace transforms of the reference, output and error signals, respectively. The plant is modelled by the transfer function

$$G(s) = \frac{1}{(s+0.5)(s+1)(s+1.5)}$$

and $K(s)$ denotes the transfer function of a compensator to be designed.

The Ziegler-Nichols tuning rule is summarised as follows:

- Apply a proportional compensator and adjust the gain until the closed-loop becomes marginally stable. Let K_{po} be the value of this gain and T_o be the period of oscillations.
- The compensator is defined by either:

(i) P: $K(s) = 0.5K_{po}$.

(ii) PI: $K(s) = 0.45K_{po} + \dfrac{0.54K_{po}/T_o}{s}$.

(iii) PID: $K(s) = 0.6K_{po} + \dfrac{1.2K_{po}/T_o}{s} + 0.075K_{po}T_o s$.

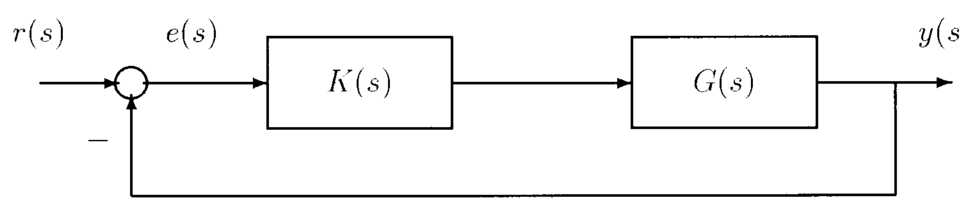
(a) Evaluate K_{po} and T_o for the feedback loop below. **[8]**

(b) Use the Ziegler-Nichols tuning rule to design a compensator $K(s)$ (either P, PI or PID) that satisfies the following specifications:

i. There is zero steady-state error against a step reference.

ii. The steady-state error against a unit ramp reference is as small as possible. **[8]**

(c) Evaluate the steady-state error due to a unit ramp reference signal for the design in Part (b). **[4]**



2. Consider the voltage feedback arrangement shown below for the speed control of a DC motor. The motor shaft drives a load with inertia J and is connected to a tacho generator. Here, v_r is the reference voltage, v_a, i_a and R_a are the armature voltage, current and resistance, respectively, v_t is the tacho voltage, w is the motor shaft speed and E is the generated EMF. Assume that

- The field flux is constant so that that E is proportional to w and the developed torque, $T(t)$, is proportional to i_a. Take the constant of proportionality to be the same and equal to k_e.
- The Power Op-Amp (POA) has negligible output resistance and dynamics, so that we can make the 'virtual earth' assumption.
- Torque disturbances and friction are negligible so that all the developed torque is supplied to the load.
- The tacho voltage is proportional to the speed with proportionality constant k_t.

In parts (a), (b) and (c) below, all references are to Laplace transforms of signals.

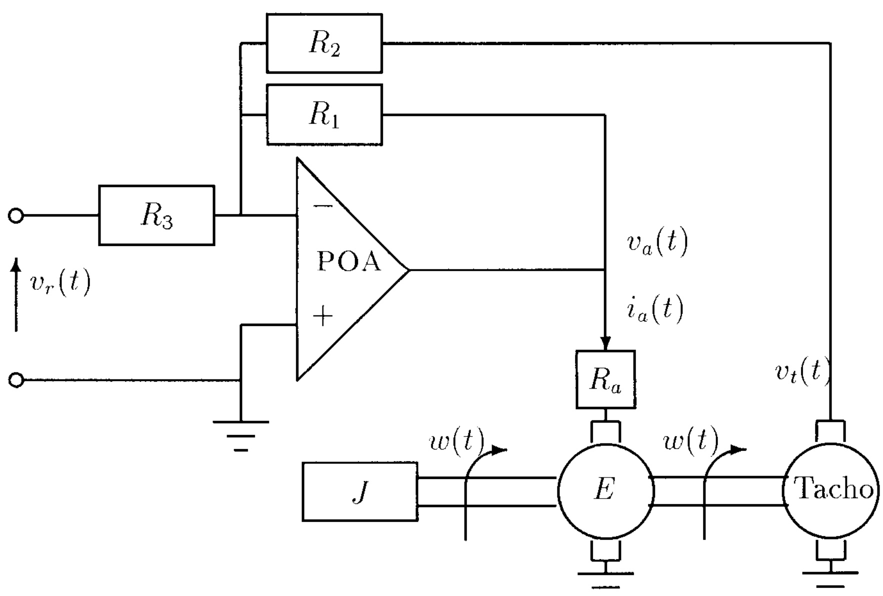
(a) Derive the transfer function $G(s) = \frac{w(s)}{v_a(s)}$. Assume zero initial conditions. **[3]**

(b) Derive an expression for $v_a(s)$ in terms of $v_r(s)$ and $w(s)$. **[3]**

(c) Hence, derive and clearly draw a block diagram representation of the feedback-loop. Take the reference signal to be $-v_r(s)$ and the output signal to be $w(s)$. Indicate clearly the signals $v_t(s)$ and $v_a(s)$ on the block diagram. **[6]**

(d) Set $R_2 = R_3 = R_a = J = k_e = k_t = 1$. Suppose that a step reference is applied at $t = 0$ of constant amplitude $-V$ (that is, $v_r(t) = -V, t \geq 0$). Find the values of V and R_1 such that

  i. The steady-state value of the shaft speed is equal to 1.
  ii. The shaft speed settles to within ±2% of its steady-state value in 3 seconds. **[8]**



3. Consider the field controlled motor illustrated in the figure below. The motor shaft has angular speed w and drives a load with inertia J and frictional damping coefficient B. The armature voltage, V_a, is assumed constant so that the developed torque, T, is proportional to the field current, i_f. Take the constant of proportionality to be K_f. The field resistance is R_f while the field inductance is L_f.

(a) Write the dynamic field loop equation (relating the field current and field voltage) and the torque balance equation (relating the angular speed and field current). (Hint: Your equations should not include $T(t)$ or V_a.) **[5]**

(b) Derive a state-variable model in the standard form:

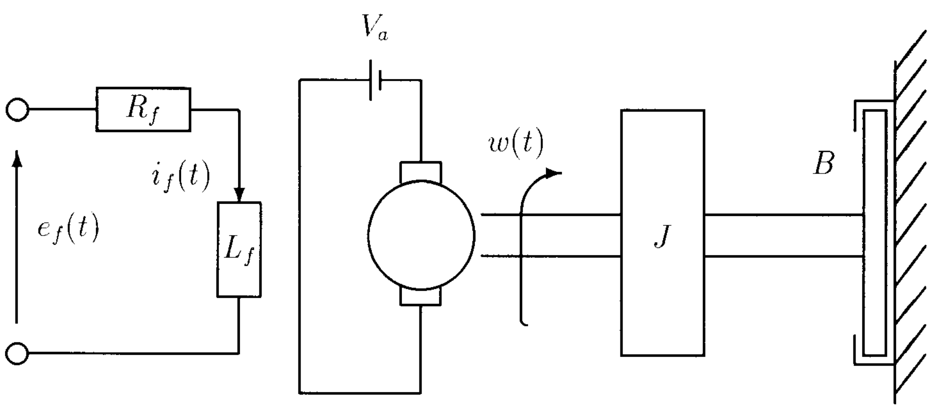
$$\begin{aligned} \dot{x}(t) &= \hat{A}x(t) + \hat{B}u(t), \\ y(t) &= \hat{C}x(t). \end{aligned}$$

Take the states to be the angular speed and the field current, the input to be the field voltage and the output to be the angular speed. **[5]**

(c) Now take $L_f = J = B = K_f = 1$.

i. Derive the transfer function between the applied field voltage and the shaft speed in terms of R_f. **[5]**

ii. Find the value of R_f so that when $e_f(t)$ is a step input, $w(t)$ reaches its steady-state value in the shortest time without overshoot. **[5]**



4. Consider the feedback loop in the figure below. Here

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

and $K(s)$ is a compensator.

(a) Take $K(s) = k$ where $k > 0$ is a constant gain. Draw the root-locus accurately as k varies in the range $0 \leq k < \infty$. Your answer should include

i. The centre and angles of the asymptotes.
ii. The range of values of k for closed-loop stability.
iii. The real-axis intercepts. Give both the closed-loop poles and the corresponding k.
iv. The imaginary-axis intercepts. What is the frequency of oscillations when the closed-loop is marginally stable? **[10]**

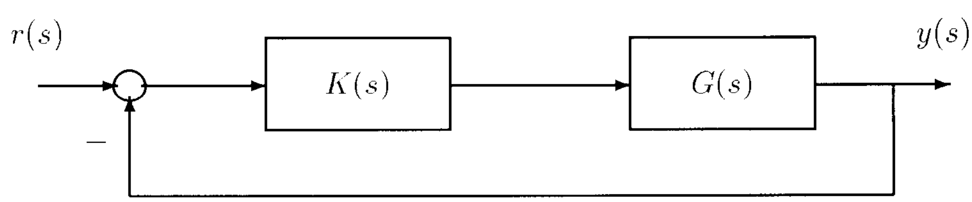
(b) Suppose that

$$K(s) = K_p + K_d s$$

is a PD compensator, where K_p and K_d are design parameters. Find K_p and K_d such that

i. the closed-loop is stable, and
ii. the two dominant poles have a damping ratio $\zeta = 1/\sqrt{2}$ and a natural frequency $\omega_n = \sqrt{2}$.

Comment on the action of the compensator on the plant. **[10]**



5. Consider the feedback control system in the figure below. Here,

$$G(s) = \frac{1}{(s+1)^3}$$

and $K(s)$ is the transfer function of a feedforward compensator.

(a) Sketch the Nyquist diagram of $G(s)$, clearly indicating the low and high frequency portions, as well as the real-axis intercepts.
[5]

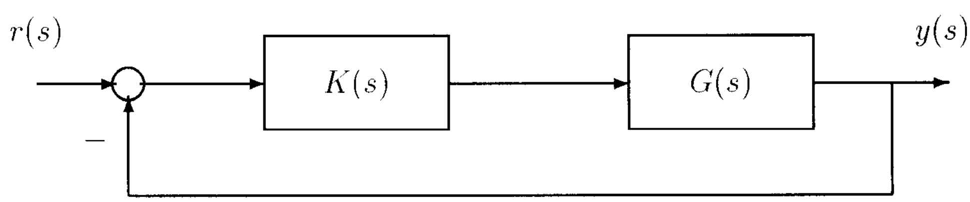
(b) Set $K(s) = K$, a constant compensator. Give the number of unstable closed-loop poles for all (positive and negative) K.
[5]

(c) Take $K = 1$. Determine the gain and phase margins.
[5]

(d) Without doing any actual design, briefly describe how a phase-lead compensator,

$$K(s) = \frac{1 + s/\omega_0}{1 + s/\omega_p}, \qquad 0 < \omega_0 < \omega_p,$$

would affect the gain and phase margins. Your answer should include a rough sketch of the Bode plots of the compensator and emphasise the difficulties involved in the design.
[5]



SOLUTIONS (E2.6/ISE2.9, Control Engineering, 2002)

1. (a) To find the critical gain K_{po} we form the Routh-Hurwitz array for the characteristic equation:

$$1+K_{po}G(s)=0 \Rightarrow s^3+3s^2+2.75s+0.75+K_{po}=0$$

$$\Longrightarrow \begin{array}{c|cc} s^3 & 1 & 2.75 \\ s^2 & 3 & 0.75+K_{po} \\ s & 2.75-(0.75+K_{po})/3 & \\ s^0 & 0.75+K_{po} & \end{array}$$

Setting the third row to zero (for marginal stability):

$$\boxed{K_{po} = 7.5.}$$

To find the critical frequency, we set the auxiliary polynomial to zero (second row with $K_{po} = 7.5$): $3s^2+(0.75+7.5)=0$. Thus $s=j\sqrt{8.25/3}$ and so

$$\boxed{T_o=2\pi/\sqrt{8.25/3}=3.7889.}$$

[8]

(b) Since we require zero steady-state error against a step reference, and since $G(s)$ is type 0, we need an integrator in the compensator. Thus we cannot use a P compensator. To ensure the smallest steady-state error against a ramp, we choose the compensator with the largest value of $\lim_{s\to 0} sK(s)$. For the PI this is $.54K_{po}/T_o$ while for the PID it is $1.2K_{po}/T_o$, which is higher. Therefore we choose the PID compensator

$$\boxed{K(s)=0.6K_{po}+1.2Kpo/sT_o+0.075K_{po}T_os=4.5+2.3754/s+2.1313s.}$$

[8]

(c) The Laplace transform of the error signal is

$$e(s) = \frac{r(s)}{1+G(s)K(s)}$$

$$\Rightarrow e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sr(s)}{1+G(s)K(s)}$$

For a unit ramp, $r(s) = 1/s^2$ and so

$$\boxed{e_{ss} = \lim_{s\to 0} \frac{1}{sG(s)K(s)} = 0.3157.}$$

[4]

2\. (a) The developed torque is $T(t) = k_e i_a(t)$ and the generated EMF is $E(t) = k_e w(t)$. Since friction is negligible and all the developed torque is supplied to the load, we have that $T(t) = J\dot{w}(t)$ or $k_e i_a(t) = J\dot{w}(t)$. However, $v_a(t) = R_a i_a(t) + E(t) = R_a i_a(t) + k_e w(t)$. It follows that $i_a(t) = \frac{1}{R_a} v_a(t) - \frac{k_e}{R_a} w(t)$.Thus

$$k_e \left(\frac{1}{R_a} v_a(t) - \frac{k_e}{R_a} w(t) \right) = J\dot{w}(t).$$

Rearranging and taking Laplace transforms (assuming zero initial conditions),

$$J\dot{w}(t) + \frac{k_e^2}{R_a} w(t) = \frac{k_e}{R_a} v_a(t) \Rightarrow \left(Js + \frac{k_e^2}{R_a} \right) w(s) = \frac{k_e}{R_a} v_a(s)$$

So

$$\boxed{G(s) = \frac{k_e/R_a}{Js + k_e^2/R_a}.}$$

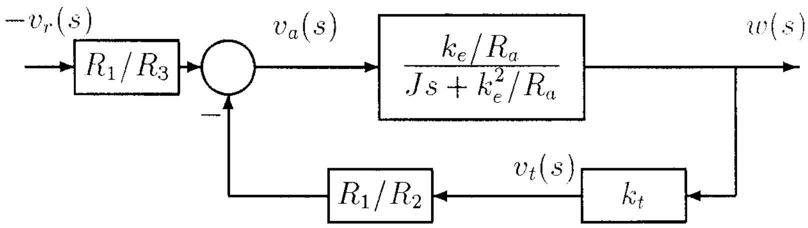
[3]

(b) Making the virtual earth assumption: $\frac{v_a(t)}{R_1} + \frac{k_t w(t)}{R_2} + \frac{v_r(t)}{R_3} = 0$, since $v_t(t) = k_t w(t)$. Taking Laplace transforms and rearranging,

$$\boxed{v_a(s) = \frac{R_1}{R_3} (-v_r(s)) - \frac{R_1}{R_2} k_t w(s).}$$

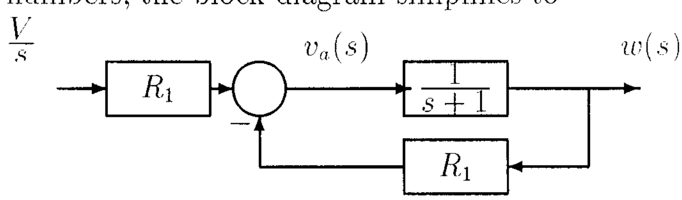
[3]

(c) Using the last equation and the expression for $G(s)$, the block diagram becomes,



[6]

(d) Putting in the numbers, the block diagram simplifies to



It follows that

$$w(s) = \frac{VR_1}{s(s+R_1+1)} = \frac{R_1 V}{R_1+1} \left(\frac{1}{s} - \frac{1}{s+R_1+1} \right) \Rightarrow w(t) = \frac{R_1 V}{R_1+1} \left(1 - e^{-(R_1+1)t} \right)$$

Since $e^{-4} < .02$, we set $3(R_1 + 1) = 4$ for a settling time of 3 seconds, or

$$\boxed{R_1 = 1/3.}$$

The steady-state value of $w(t)$ is $\frac{R_1 V}{R_1+1}$ and it follows that

$$\boxed{V = 4}$$

for a steady-state value of 1.

[8]

3. (a) The field equation is

$$\boxed{L_f \frac{d}{dt} i_f(t) + R_f i_f(t) = e_f(t).}$$

Since the applied torque is $K_f i_f(t)$, the equation for the shaft torque balance equation becomes

$$\boxed{J \frac{d}{dt} w(t) + B w(t) = K_f i_f(t).}$$

[5]

(b) Let

$$x_1(t) = w(t), \quad x_2(t) = i_f(t), \quad u(t) = e_f(t) \ \ \& \ y(t) = w(t).$$

Then the above equations can be written as

$$\boxed{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{B}{J} & \frac{K_f}{J} \\ 0 & -\frac{R_f}{L_f} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_f} \end{bmatrix} u(t),}$$

and

$$\boxed{y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.}$$

[5]

(c) i. By either taking the Laplace transforms in Part (a) and eliminating $i_f(s)$ or using $G(s) = C(sI - A)^{-1}B$ in Part (b), we get

$$\boxed{\frac{w(s)}{e_f(s)} = G(s) = \frac{1}{(s+1)(s+R_f)}.}$$

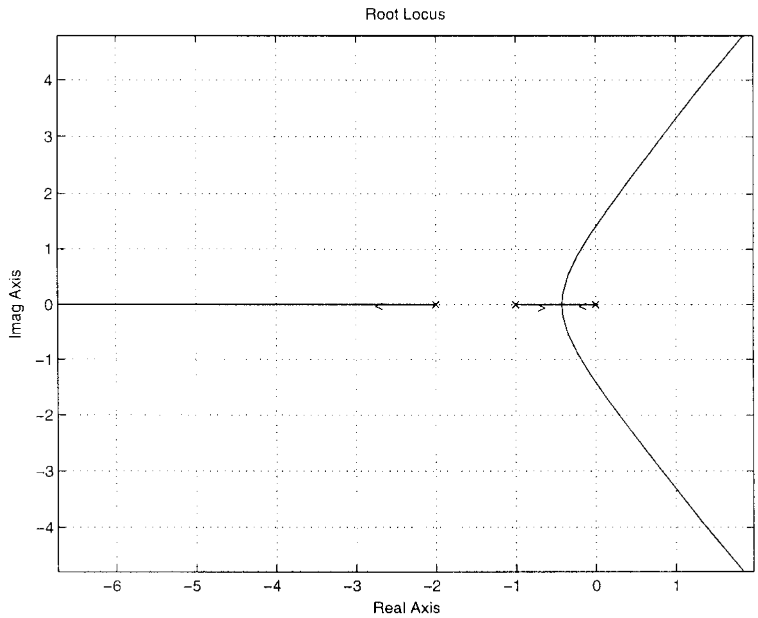
[5]

ii. For the response to reach its steady-state value in least time without overshoot, the system must be critically damped, that is, the poles must be real and equal. It follows that

$$\boxed{R_f = 1.}$$

[5]

4. (a) The root-locus plot is shown below.



i. The centre and angles of the asymptotes are $\boxed{\sigma = -1 \ \& \ \psi = \pm 60^o, 180^o.}$

ii. The characteristic equation and the Routh array are

$$s^3+3s^2+2s+k=0 \implies \begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \frac{6-k}{3} & \\ s^0 & k & \end{array}$$

The range of k for closed-loop stability is therefore $\boxed{0 < k < 6.}$

iii. The imaginary axis intercept corresponds to $\boxed{k = 6.}$ To find the associated critical frequency, we set the auxiliary polynomial to zero (second row with $k = 6$): $3s^2 + 6 = 0$, or $s = j\omega$ where $\boxed{\omega = \sqrt{2}.}$

iv. For the real axis intercepts, we search for real roots of $\frac{d}{ds}\frac{1}{G(s)} = 3s^2 + 6s + 2 = 0$. This gives $s = -1 \pm 1/\sqrt{3}$ and so the real axis intercept is $\boxed{s = -1 + 1/\sqrt{3}.}$ The corresponding value of k is found from the gain criterion: $\boxed{k = -1/G(-1 + 1/\sqrt{3}) = 2.5981.}$ **[10]**

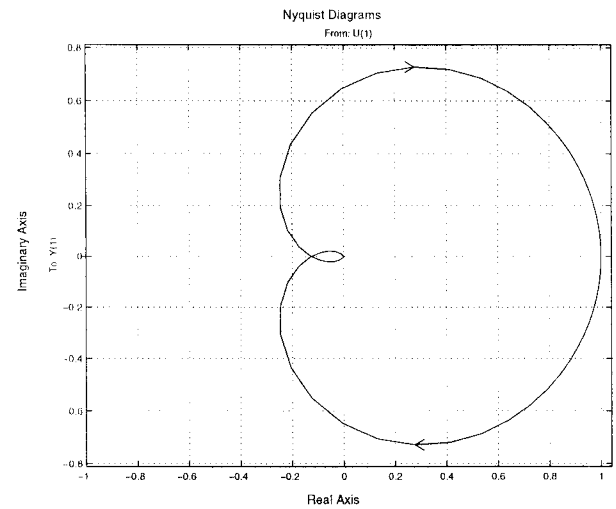
(b) Write $K(s) = K_p + K_d s = K_d(s - z)$ where $z = -K_p/K_d$. The required location of the dominant poles is $p, \bar{p} = -1 \pm j$ to give $\zeta = 1/\sqrt{2}$ and $\omega_n = \sqrt{2}$. To find the value of z we use the angle criterion: $\angle(p - 0) + \angle(p + 1) + \angle(p + 2) - \angle(p - z) = 180^o$ or $135^o + 90^o + 45^o - \theta = 180^o$ or $\theta = 90^o$. Thus $z = -1$. To find the corresponding gain, we use the gain criterion: $K_d = -\frac{1}{G(p)(p - z)} = -\frac{p(p + 1)(p + 2)}{(p + 1)} = -p(p + 2) = 2$. Thus $\boxed{K_d = 2 \ \& \ K_p = -K_d z = 2.}$

$\boxed{\text{The compensator has cancelled one of the poles of the plant.}}$ **[10]**

5. (a) The Nyquist plot is shown below. The real-axis intercepts are found by setting $Im[G(j\omega)]=0$. Thus $\boxed{\omega_i=0, \pm\sqrt{3}, \infty}$ so $\boxed{G(j\omega_i)=1, -0.125, -0.125, 0.}$ **[5]**

(b) The number of unstable closed-loop poles associated with gain K can be determined by the number of encirclements by $G(s)$ of the point $-1/K$. Thus $\boxed{0<K<8 \Rightarrow \text{stable},\ K>8 \Rightarrow 2\,\text{unstable},\ -1<K<0 \Rightarrow \text{stable},\ K<-1 \Rightarrow 1\ \text{unstable}}$ **[5]**

(c) Since the negative real-axis intercept is at -0.125, then $\boxed{\text{the gain margin is 8.}}$ For the phase margin we solve $|G(j\omega)|=1$. However, the Nyquist diagram is inside the unit circle except when $w=0$. Thus: $\boxed{\text{the phase margin is } 180^o.}$ **[5]**



(d) The phase-lead has gain close to unity for frequencies below ω_0 and close to $\frac{\omega_p}{\omega_0} > 1$ beyond ω_p. The phase is positive and large between $\omega_0 \& \omega_p$ but small below and above. The Bode plots are shown below. The increase in gain at frequencies above ω_p tends to degrade the stability margins, while the phase-lead tends to increase the phase margin, which is stabilising. It is thus important to balance the destabilising increase in gain against the stabilising increase in phase. $\boxed{\text{We should place } w_p \text{ and } w_0 \text{ in the crossover frequency range } (|G(j\omega)| \approx 1.)}$ **[5]**