**PROBLEM 1**

(a) The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answer.

(i) $h[n] = (\frac{1}{2})^n u[n]$

(ii) $h[n] = (0.6)^n u[n+2] + (0.5)^n u[-n]$

(iii) $h[n] = 2^n u[3-n]$

(b) Consider the first-order difference equation

$$y[n] + 2y[n-1] = x[n]$$

Assume the condition of **initial rest**. This means that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$. Find the impulse response of a causal system whose input and output are related by this difference equation. Assume that $x[n] = 0$ for $n < 0$.

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**PROBLEM 2**

(a) Let $x[n]$ be a discrete periodic signal with period $N$ whose Fourier series coefficients are $a_k$ with period $N$. Determine the Fourier series coefficients of the signal $y[n] = x[n] - x[n-1]$.

(b) Let

$$g[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & 6 \le n \le 7 \end{cases}$$

be a periodic signal with fundamental period $N = 8$. Determine the Fourier series coefficients of the signal $g[n]$.

(c) Consider the signal $w[n] = g[n] - g[n-1]$ with $g[n]$ as defined in (b).

(i) Determine the Fourier series coefficients of the signal $w[n]$ using the definition.

(ii) Determine the Fourier series coefficients of the signal $w[n]$ using the result of (a).

Use the relationship $\sum_{i=0}^{N-1} x^i = \frac{1-x^N}{1-x}, \ |x| \le 1$.

**PROBLEM 3**

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$$

Determine the frequency response of the system and then find and sketch its Bode plots.

**PROBLEM 4**

(a) Consider a continuous time LTI system. Prove that the response of the system to a complex exponential input $e^{s_0 t}$ is the same complex exponential with only a change in amplitude; that is $H(s_0)e^{s_0 t}$. The function $H(s)$ is the Laplace transform of the impulse response of the system.

(b) A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t) = e^t$ for all $t$, the output is $y(t) = \frac{11}{12}e^t$.

2. When the input to the system is $x(t) = e^{2t}$ for all $t$, the output is $y(t) = \frac{7}{10}e^{2t}$.

3. The impulse response $h(t)$ satisfies the equation

$$h(t) = ae^{-3t}u(t) + be^{-2t}u(t)$$

where $a, b$ are unknown constants.

Determine the response $H(s)$ of the system, consistent with the information above. The constants $a, b$ should not appear in your answer.

Use the fact that the Laplace transform of the function $h(t) = e^{-at}u(t)$ is $H(s) = \frac{1}{s+a}$, $\mathrm{Re}\{s\} > -a$.

**PROBLEM 5**

(a) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete causal signal $x[n] = (\frac{1}{2})^n u[n]$, with $u[n]$ the discrete unit step function.

Use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

(b) Consider the causal LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

(i) Determine the z-transform of the impulse response.

(ii) Determine the z-transform of the output if $x[n] = (\frac{1}{2})^n u[n]$.

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**Answer 1**

(a) (i) $h[n] = (\frac{1}{2})^n u[n]$. The system is causal since $h[n] = 0$ for $n < 0$ and stable since $\lim_{n\to+\infty} h[n] = 0$.

(ii) $h[n] = (0.6)^n u[n+2] + (0.5)^n u[-n]$. The system is non-causal because of the term $(0.5)^n u[-n]$ and non-stable since $\lim_{n\to+\infty} h[n] = +\infty$.

(iii) $h[n] = 2^n u[3-n]$. The system is non-causal and stable since $\lim_{n\to+\infty} h[n] = 0$.

(b) Consider the first-order difference equation $y[n] + 2y[n-1] = x[n] \Rightarrow y[n] = -2y[n-1] + x[n]$. From this we get

$y[0] = -2y[-1] + d[0] = 1$

$y[1] = -2y[0] = -2$

$y[2] = -2y[1] = (-2)(-2) = 4$

$y[3] = -2y[2] = -6$

$\vdots$

$y[n] = (-2)^n,\ n \geq 0 \Rightarrow y[n] = (-2)^n u[n]$

**Answer 2**

(a) The signal $x[n]$ is written using the Fourier series representation as follows:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

From the above we have

$$x[n-1] = \sum_{k=\langle N \rangle} a_k e^{jkw_0(n-1)} = \sum_{k=\langle N \rangle} a_k e^{jk(2p/N)(n-1)} = \sum_{k=\langle N \rangle} e^{-jk(2p/N)} a_k e^{jk(2p/N)n}$$

Thus,

$$y[n] = x[n] - x[n-1] = \sum_{k=\langle N \rangle} (1 - e^{-jk(2p/N)}) a_k e^{jk(2p/N)n}$$

Hence, the Fourier series coefficients of the signal $y[n] = x[n] - x[n-1]$ are

$$b_k = (1 - e^{-jk(2p/N)}) a_k$$

(b) The Fourier series coefficients of the signal $g[n]$ are given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2p/N)n} = \frac{1}{N} \sum_{n=0}^{7} g[n] e^{-jk(2p/N)n} = \frac{1}{8} \sum_{n=0}^{5} e^{-jk(p/4)n} = \frac{1}{8} \frac{1 - e^{-jk(3p/2)}}{1 - e^{-jk(p/4)}}$$

(c) (i) The signal $w[n]$ is also periodic with period $N = 8$ and is given by

$w[0] = g[0] - g[-1] = 1$
$w[1] = g[1] - g[0] = 0$
$w[2] = g[2] - g[1] = 0$
$w[3] = g[3] - g[2] = 0$
$w[4] = g[4] - g[3] = 0$
$w[5] = g[5] - g[4] = 0$
$w[6] = g[6] - g[5] = -1$
$w[7] = g[7] - g[6] = 0$

The Fourier series coefficients of the signal $w[n]$ are given by

$$b_k = \frac{1}{8} \sum_{n=0}^{7} w[n] e^{-jk(p/4)n} = \frac{1}{8}(1 - e^{-jk(3p/2)})$$

(ii) According to the result of Part (a) and provided that the Fourier series coefficients of the signal $g[n]$ are given by

$$a_k = \frac{1}{8} \frac{1 - e^{-jk(3p/2)}}{1 - e^{-jk(p/4)}}$$

the Fourier series coefficients of the signal $w[n]$ are given by

$$b_k = (1 - e^{-jk(p/4)}) \frac{1}{8} \frac{1 - e^{-jk(3p/2)}}{1 - e^{-jk(p/4)}} = \frac{1}{8}(1 - e^{-jk(3p/2)})$$

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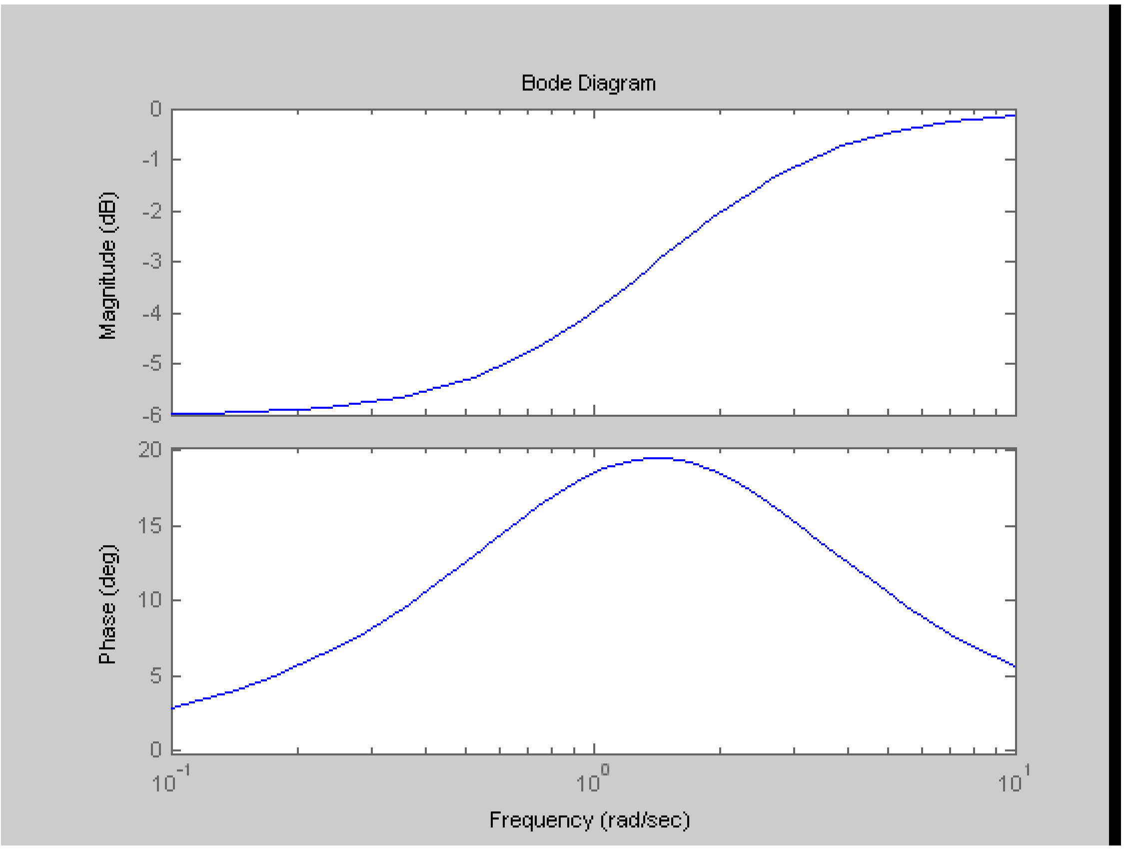
**Answer 3**

We first have to find the frequency response of the system.

From $\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$ if we take the Fourier transform in both sides we get $Y(jw)[(jw) + 2] = X(jw)[(jw) + 1] \Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{jw + 1}{jw + 2}$. You can treat this function easily since for the Bode plots of $H(jw)$ you need to find the Bode plots of the functions $jw + 1$ and $jw + 2$ and subtract them.




**Answer 4**

(a) The output of the system $y(t)$ is given as the convolution between the input of the system $x(t)=e^{s_0 t}$ and the impulse response $h(t)$. This will be

$$y(t)=\int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s_0(t-\tau)}h(\tau)d\tau = e^{s_0 t}\int_{-\infty}^{+\infty} e^{-s_0 \tau}h(\tau)d\tau = e^{s_0 t}H(s_0)$$

where $H(s)$ is the Laplace transform of the impulse response given by $H(s)=\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$ evaluated at $s=s_0$.

(b) The Laplace transform of the impulse response $h(t)=ae^{-3t}u(t)+be^{-2t}u(t)$ is $H(s)=\frac{a}{s+3}+\frac{b}{s+2}$, $\mathrm{Re}\{s\}>-2$. According to the information provided we have that $H(1)=\frac{11}{12}$ and $H(2)=\frac{7}{10}$. We form the system of equations:

1. $H(1)=\frac{a}{4}+\frac{b}{3}=\frac{11}{12}$
2. $H(2)=\frac{a}{5}+\frac{b}{4}=\frac{7}{10}$

From (1) and (2) we have $a=1, b=2$. Thus,

$$H(s)=\frac{1}{s+3}+\frac{2}{s+2}=\frac{3s+8}{(s+3)(s+2)},\quad \mathrm{Re}\{s\}>-2$$

**Answer 5**

(a) Consider the function

$$x[n]=\begin{cases}(\frac{1}{2})^n & n\geq 0\\ 0 & n<0\end{cases}$$

The z-transform expression is

$$X(z)=1+(\frac{1}{2})z^{-1}+(\frac{1}{2})^2z^{-2}+(\frac{1}{2})^3z^{-3}+\ldots=1+(\frac{1}{2}z^{-1})+(\frac{1}{2}z^{-1})^2+(\frac{1}{2}z^{-1})^3+\ldots\Rightarrow$$

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\ |\frac{1}{2}z^{-1}|\leq 1$$

(b) (i) By taking the z-transform in both sides of the input-output relationship we end up with the following expression for the z-transform of the system.

$$Y(z)-z^{-1}Y(z)+\frac{1}{4}z^{-2}Y(z)=X(z)-\frac{1}{2}z^{-1}X(z)\Rightarrow\frac{Y(z)}{X(z)}\Rightarrow H(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\ |\frac{1}{2}z^{-1}|\leq 1$$

(ii) $Y(z)=H(z)X(z)=\dfrac{1}{(1-\frac{1}{2}z^{-1})^2},\ |\frac{1}{2}z^{-1}|\leq 1$