IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2003

SIGNALS AND LINEAR SYSTEMS

Monday, 2 June 2:00 pm

Time allowed: 2:00 hours

There are FIVE questions on this paper.

Answer THREE questions.

Corrected Copy

Any special instructions for invigilators and information for candidates are on page 1.

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Second Marker(s) : A.G. Constantinides

1.

Consider the cascade interconnection of three linear time invariant (LTI) systems, as in Figure 1.1. The impulse response $h_1[n]$ is

$$h_1[n] = \delta[n-3]$$

and the impulse response $h_2[n]$ is

$$h_2[n] = \delta[n-5]$$

where $\delta[n]$ is the discrete unit impulse function defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

The impulse response $h_3[n]$ is unknown.
The overall impulse response is as shown in Figure 1.2.

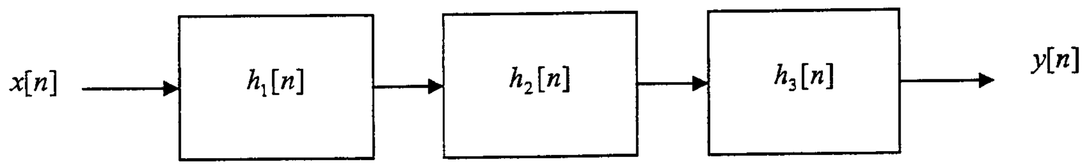


Figure 1.1

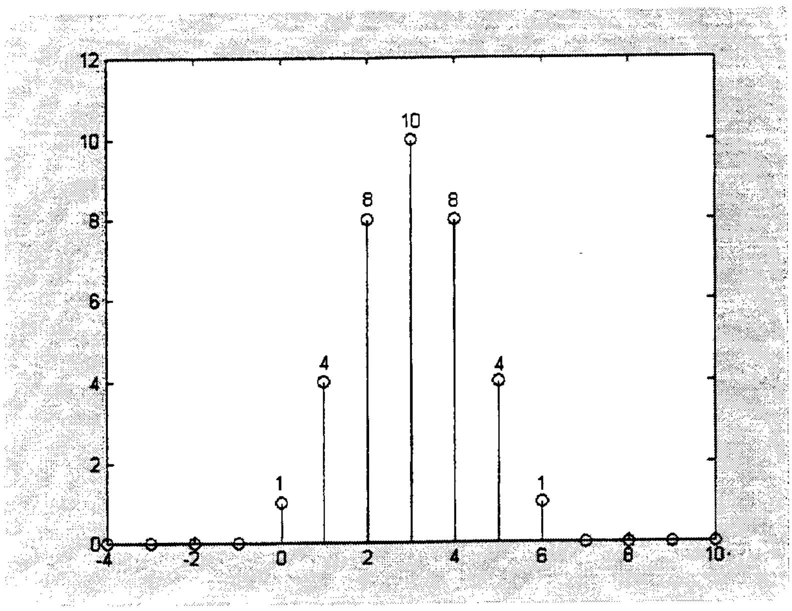


Figure 1.2

(a) Find the impulse response $h_3[n]$. **[10]**

(b) Find the output of the overall system to the input

$$x[n] = \delta[n-1] - \delta[n-2]$$

where $\delta[n]$ is as defined above.

[10]

2.

(a) Consider the discrete, real and odd signal $x_1[n]$ that is periodic with period $N=7$ and fundamental frequency $\omega_0 = \frac{2\pi}{N}$. The Fourier series coefficients c_k of $x_1[n]$ are odd, purely imaginary and given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]e^{-jk(2\pi/N)n}$$

(i) Show that c_k are periodic with respect to k with period $N=7$.

[2]

(ii) Given that

$$c_{15} = j,\ c_{16} = 2j,\ c_{17} = 3j$$

determine the values of c_0, c_{-1}, c_{-2}, c_{-3}.

[5]

(b) Find the Fourier Transform of the discrete signal $x_2[n] = a^n u[n], |a| < 1$, where $u[n]$ is the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

You may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

[5]

(c) Consider the discrete signal $x_3[n]$ that is aperiodic. Find the Fourier transform of the signal $y[n] = x_3[n-n_0]$ where n_0 is a constant.

[3]

(d) The input $x[n]$ and output $y[n]$ of a stable and causal LTI system are related by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

Find the impulse response of this system.

[5]

3.

The output $y(t)$ of a LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=\frac{dx(t)}{dt}+2x(t)$$

(a) Determine the transfer function of the system and sketch the Bode plots associated with it.

[13]

(b) If $x(t)=e^{-2t}u(t)$, determine the output of the system in the frequency domain.

[7]

4.

(a) Consider a continuous time LTI system. Assume the input to the system to be $x(t) = e^{s_0 t}$, where s_0 is a constant and show that if the transfer function of the system is $H(s)$ then the output is $e^{s_0 t} H(s_0)$.

[7]

(b) A causal LTI system with impulse response $g(t)$ has the following properties:

1. When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{1}{6} e^{2t}$ for all t.
2. The impulse response $g(t)$ satisfies the differential equation:

$$\frac{dg(t)}{dt} + 2g(t) = e^{-4t}u(t) + bu(t)$$

where b is an unknown constant.

Determine the Laplace transform $G(s)$ of the impulse response of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in your answer.

[13]

5.

(a) (i) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete causal signal $x[n] = a^n u[n]$, with a real and $u[n]$ the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

[3]

(ii) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete anti-causal signal $x[n] = -a^n u[-n-1]$, with a real and $u[n]$ the discrete unit step function.

[3]

For parts a(i)-a(ii) you may wish to use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

(b) Consider a LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n-2] - \frac{5}{2} y[n-1] + y[n] = x[n]$$

Determine the impulse response and its z-transform in the following three cases:

(i) The system is causal.

(ii) The system is stable.

(iii) The system is neither stable nor causal.

[14]

1.

(a) Find the impulse response $h_3[n]$.

$h_1[n] * h_2[n] = \delta[n-8]$

$\delta[n-8] * h_3[n] = \delta[n] + 4\delta[n-1] + 8\delta[n-2] + 10\delta[n-3] + 8\delta[n-4] + 4\delta[n-5] + \delta[n-6] \Rightarrow$

$h_3[n] = \delta[n+8] + 4\delta[n+7] + 8\delta[n+6] + 10\delta[n+5] + 8\delta[n+4] + 4\delta[n+3] + \delta[n+2]$

(b) Find the output of the overall system to the input

$$x[n] = \delta[n-1] - \delta[n-2]$$

where $\delta[n]$ is the discrete unit impulse function defined above.

$(\delta[n] - \delta[n-1]) * (\delta[n] + 4\delta[n-1] + 8\delta[n-2] + 10\delta[n-3] + 8\delta[n-4] + 4\delta[n-5] + \delta[n-6]) =$

$\delta[n] + 4\delta[n-1] + 8\delta[n-2] + 10\delta[n-3] + 8\delta[n-4] + 4\delta[n-5] + \delta[n-6] -$

$(\delta[n-1] + 4\delta[n-2] + 8\delta[n-3] + 10\delta[n-4] + 8\delta[n-5] + 4\delta[n-6] + \delta[n-7]) =$

$\delta[n] + 3\delta[n-1] + 4\delta[n-2] + 2\delta[n-3] - 2\delta[n-4] - 4\delta[n-5] - 3\delta[n-6] - \delta[n-7])$

2.

(a) Consider the discrete, real and odd signal $x[n]$ that is periodic with period $N = 7$ and fundamental frequency $\omega_0 = \frac{2\pi}{N}$. Suppose that the Fourier series coefficients of $x(t)$ are c_k.

(i)

$$c_{k+N} = \frac{1}{N}\sum_{n-0}^{N-1} x_1[n]e^{-j(k+N)(2\pi/N)n} = \frac{1}{N}\sum_{n-0}^{N-1} x_1[n]e^{-jk(2\pi/N)n}e^{-j2\pi n} = c_k$$

(ii) Since the signal is real and odd, the Fourier series coefficients c_k will be purely imaginary and odd. Given that

$$c_{15} = j,\ c_{16} = 2j,\ c_{17} = 3j$$

determine the values of c_0, c_{-1}, c_{-2}, c_{-3}.

$c_0 = 0$

$c_{-1} = -c_1 = -c_{15} = -j$

$c_{-2} = -c_2 = -c_{16} = -2j$

$c_{-3} = -c_3 = -c_{16} = -3j$

(b) Find the Fourier transform of the discrete signal $x[n] = a^n u[n], |a| < 1$.

In this case $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}},\ |a| < 1$

(c) Consider the discrete signal $x[n]$ that is aperiodic. Find the Fourier transform of the signal $y[n] = x[n - n_0]$.

In this case $\sum_{n=-\infty}^{+\infty} x[n - n_0]e^{-j\omega(n-n_0)}e^{-j\omega n_0} = e^{-j\omega n_0} X(e^{j\omega})$

(d) The input and output of a stable and causal LTI system are related by the differential equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

Find the impulse response of this system.

We take the Fourier transform in both sides:

$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-2j\omega}Y(e^{j\omega}) = 2X(e^{j\omega}) \Rightarrow$

$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{2}{(1 - \frac{1}{4}e^{-j\omega})(1 - \frac{1}{2}e^{-j\omega})} =$

$$\frac{2[2(1-\frac{1}{4}e^{-j\omega})-(1-\frac{1}{2}e^{-j\omega})]}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}=\frac{4}{(1-\frac{1}{2}e^{-j\omega})}-\frac{2}{(1-\frac{1}{4}e^{-j\omega})}\Rightarrow$$

$$h[n]=[4(\frac{1}{2})^n-2(\frac{1}{4})^n]u[n]$$

3.

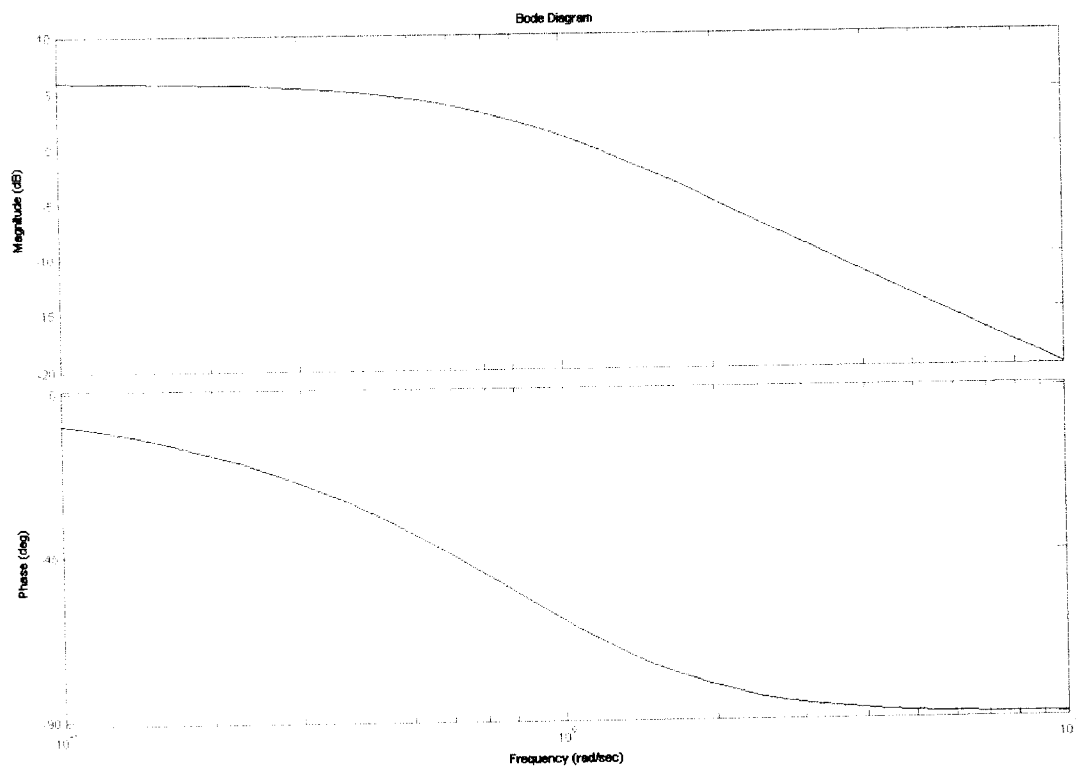
The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=\frac{dx(t)}{dt}+2x(t)$$

(a) Determine the frequency response of the system, then find and sketch its Bode plots.

$$(j\omega)^2Y(e^{j\omega})+2j\omega Y(e^{j\omega})+Y(e^{j\omega})=j\omega X(e^{j\omega})+2X(e^{j\omega})\Rightarrow$$

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{j\omega+2}{(j\omega)^2+2j\omega+1}=\frac{j\omega+2}{(j\omega+1)^2}$$



(b) If $x(t)=e^{-2t}u(t)$, determine the output of the system in the frequency domain.

$$X(e^{j\omega})=\frac{1}{2+j\omega}\Rightarrow Y(e^{j\omega})=\frac{1}{(j\omega)^2+2j\omega+1}$$

4.

(a) Consider a continuous time LTI system. Find the response of the system to a complex exponential input $e^{s_0 t}$, as a function of the Laplace transform of the impulse response of the system.

$$y(t)=\int_{-\infty}^{+\infty} h(\tau)e^{s_0(t-\tau)}d\tau = e^{s_0 t}\int_{-\infty}^{+\infty} h(\tau)e^{-s_0\tau}d\tau = e^{s_0 t}H(s_0)$$

(b) A causal LTI system with impulse response $h(t)$ has the following properties:

1. When the input to the system is $x(t)=e^{2t}$ for all t, the output is $y(t)=\frac{1}{6}e^{2t}$ for all t.
2. The impulse response satisfies the differential equation:

$$\frac{dh(t)}{dt}+2h(t)=e^{-4t}u(t)+bu(t)$$

where b is an unknown constant.

Determine the Laplace transform of the impulse response of the system $H(s)$, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in your answer.

$$\frac{dh(t)}{dt}+2h(t)=e^{-4t}u(t)+bu(t)\Rightarrow (s+2)H(s)=\frac{1}{s+4}+\frac{b}{s}\Rightarrow$$

$$H(s)=\frac{s+b(s+4)}{(s+2)(s+4)s}$$

$$H(2)=\frac{2+6b}{48}=\frac{1}{6}\Rightarrow\frac{2+6b}{8}=1\Rightarrow b=1\Rightarrow H(s)=\frac{2}{(s+4)s}$$

5.

(a) (i) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete causal signal $x[n]=a^n u[n]$, with a real and $u[n]$ the discrete unit step function.

$$X(z)=\frac{z}{z-a},\ |z|>|a|$$

(ii) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete anti-causal signal $x[n]=-a^n u[-n-1]$, with a real and $u[n]$ the discrete unit step function.

$$X(z)=\frac{z}{z-a},\ |z|<|a|$$

(b) Determine the impulse response and the z-transform of the impulse response, for the LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n-2]-\frac{5}{2}y[n-1]+y[n]=x[n]$$

$$y[n-2]-\frac{5}{2}y[n-1]+y[n]=x[n]\Rightarrow (z^{-2}-\frac{5}{2}z^{-1}+1)Y(z)=X(z)\Rightarrow$$

$$\frac{Y(z)}{X(z)}=\frac{1}{z^{-2}-\frac{5}{2}z^{-1}+1}=\frac{1}{(z^{-1}-2)(z^{-1}-\frac{1}{2})}=\frac{\frac{2}{3}[(z^{-1}-\frac{1}{2})-(z^{-1}-2)]}{(z^{-1}-2)(z^{-1}-\frac{1}{2})}=\frac{\frac{2}{3}}{(z^{-1}-2)}-\frac{\frac{2}{3}}{(z^{-1}-\frac{1}{2})}$$

Impulse response that corresponds to the term $\dfrac{\frac{2}{3}}{(z^{-1}-2)} = \dfrac{2}{3}\dfrac{z}{1-2z} = \dfrac{1}{3}\dfrac{z}{1/2-z} = (-\dfrac{1}{3})\dfrac{z}{z-1/2}$ can be

- $h_1[n] = (-\frac{1}{3})(\frac{1}{2})^n u[n]$ or
- $h_2[n] = \frac{1}{3}(\frac{1}{2})^n u[-n-1]$

Impulse response that corresponds to the term $\dfrac{-\frac{2}{3}}{(z^{-1}-\frac{1}{2})} = (-\dfrac{2}{3})\dfrac{z}{1-\frac{1}{2}z} = \dfrac{2}{3}\dfrac{z}{\frac{1}{2}z-1} = \dfrac{4}{3}\dfrac{z}{z-2}$ can be

- $h_3[n] = \frac{4}{3}2^n u[n]$ or
- $h_4[n] = -\frac{4}{3}2^n u[-n-1]$

In order for the system to be stable:

$$h[n] = (-\frac{1}{3})(\frac{1}{2})^n u[n] - \frac{4}{3}2^n u[-n-1]$$

In order for the system to be causal:

$$h[n] = (-\frac{1}{3})(\frac{1}{2})^n u[n] + \frac{4}{3}2^n u[n]$$

In order for the system to be neither stable not causal the remaining two combinations should be considered.