Paper Number(s): **E2.5**
ISE2.7

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2002

EEE/ISE PART II: M.Eng., B.Eng. and ACGI

SIGNALS AND LINEAR SYSTEMS

Monday, 27 May 2:00 pm

There are FIVE questions on this paper.

Answer THREE questions.

Corrected copy

Time allowed: 2:00 hours

Examiners responsible:

$\rightarrow$ Q4 (a)

First Marker(s): Stathaki,T.

Second Marker(s): Constantinides,A.G.

1.

Consider the cascade interconnection of three causal linear time invariant (LTI) systems, illustrated in the following Figure 1. The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-3],$$

where $u[n]$ is the discrete unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The overall impulse response is as shown in Figure 2.

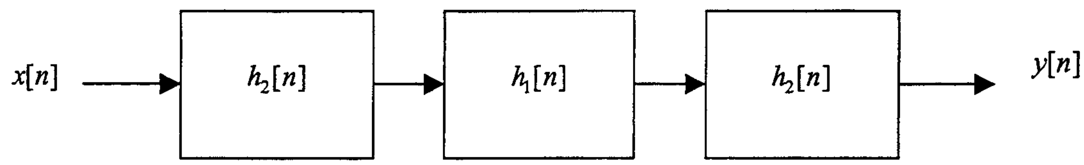

Figure 1

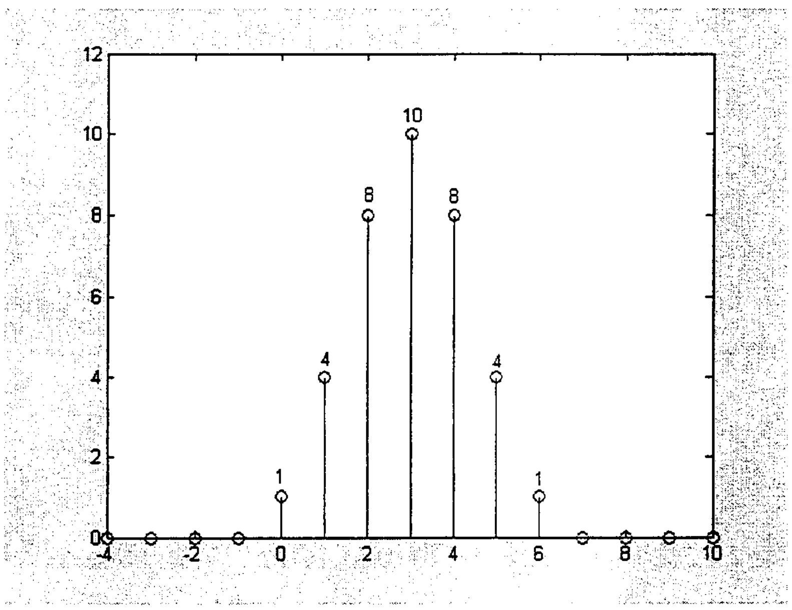

Figure 2

(a) Show that the convolution of $h_2[n]$ with itself is given by

$$h_2[n] * h_2[n] = \delta[0] + 2\delta[1] + 3\delta[2] + 2\delta[3] + \delta[4]$$

[6]

(b) Find the impulse response $h_1[n]$. **[7]**

(c) Find the output of the overall system to the input

$$x[n] = \delta[n-1] - \delta[n-3]$$

where $\delta[n]$ is the discrete unit impulse function defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

[7]

For part (a) use the fact that given two discrete signals $x[n]$ and $y[n]$ with finite durations of M and N samples respectively, the convolution $x[n] * y[n]$ is of duration $M + N - 1$ samples.

2.

(a) Consider the signal $x(t)$ that is periodic with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Suppose that the Fourier series coefficients of $x(t)$ are c_k. Find the Fourier series coefficients of the signal $y(t) = \frac{dx(t)}{dt}$. **[2]**

(b) Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$c_k = \begin{cases} 1, & k = 0 \\ -j\left(\frac{1}{3}\right)^{|k|}, & \text{otherwise} \end{cases}$$

(i) Is $x(t)$ real? **[3]**

(ii) Is $x(t)$ odd? **[3]**

(iii) Is $\frac{dx(t)}{dt}$ odd? **[3]**

Justify your answers.

(c) Consider the signal $w(t)$ that is aperiodic. Find the Fourier transform of the signal $y(t) = \frac{dw(t)}{dt}$. **[2]**

(d) Find the Fourier transform of the signal $v(t) = e^{-at}u(t)$. Assume that the real part of a is positive and that $u(t)$ is the continuous unit step function defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

[2]

(e) The input and output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2x(t)$$

(i) Find the impulse response of this system. **[2]**

(ii) What is the frequency response of the output of this system if $x(t) = e^{-3t}u(t)$? **[3]**

3.

The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=x(t)$$

(a) Determine the frequency response of the system, then find and sketch its Bode plots. [13]

(b) If $x(t)=e^{-2t}u(t)$, determine the output of the system in the frequency domain. [7]

4.

(a) Consider an LTI system with input $x(t) = e^{-t}u(t)$ and impulse response $h(t) = e^{-3t}u(t)$.

(i) Determine the Laplace transforms of $x(t)$ and $h(t)$. **[3]**

(ii) From (i) find the Laplace transform $Y(s)$ of the output $y(t)$ of the system. **[3]**

(iii) From $Y(s)$ as obtained in part (ii) determine $y(t)$. **[3]**

(iv) Verify your result in part (iii) by explicitly convolving $x(t)$ and $h(t)$. **[3]**

(b) (i) Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ and Laplace transform $X(s) = s + 1$, $\Re e\{s\} < -1$, with $\Re e\{s\}$ the real part of s. Draw the pole-zero plot for $X(s)$ on the s-plane. Also, draw the vector whose length represents $|X(j\omega)|$ and whose angle with respect to the real axis represents $\angle X(j\omega)$ for a given ω. **[3]**

(ii) Repeat (i) for a signal $y(t)$ with Fourier transform $Y(j\omega)$ and Laplace transform $Y(s) = s - 1$, $\Re e\{s\} < 1$. **[3]**

(iii) By using the results of parts b-(i) and b-(ii) compare the amplitudes $|X(j\omega)|$ and $|Y(j\omega)|$. Also, compare the phases $\angle X(j\omega)$ and $\angle Y(j\omega)$. **[2]**

5.

(a) (i) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete causal signal $x[n] = a^n u[n]$, with a real and $u[n]$ the discrete unit step function.

[3]

(ii) Find the analytical expression and the region of convergence (ROC) of the z-transform of the discrete anti-causal signal $x[n] = -a^n u[-n-1]$, with a real and $u[n]$ the discrete unit step function. **[3]**

(iii) Is the analytical expression $X(z)$ of the z-transform of a signal sufficient in order to determine the analytical expression $x[n]$ of the signal in time? **[3]**

For parts a(i)-a(ii) use the relationship $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$, if $|x| < 1$.

(b) Consider a discrete causal signal $x[n]$, with $x[n] = 0$ for $n < 0$. Find the z-transform of the signal $x[n-m],\ m > 0$ as a function of the z-transform of the signal $x[n]$. **[3]**

(c) Determine the impulse response and the z-transform of the impulse response, for the LTI system with input $x[n]$ and output $y[n]$ related with the difference equation

$$y[n] - \frac{5}{4}y[n-1] + \frac{3}{8}y[n-2] = x[n]$$

in the following two cases:

(i) The system is causal **[4]**

(ii) The system is anti-causal **[4]**

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Problem 1

(a) $h_2[n] * h_2[n] = u[n] * u[n] - 2u[n] * u[n-3] + u[n-3] * u[n-3]$

$= (n+1)u[n] - 2(n-2)u[n-3] + (n-5)u[n-6] \Rightarrow$

$$g[n] = h_2[n] * h_2[n] = \begin{cases} n+1, & n = 0,1,2 \\ 5-n, & n = 3,4,5 \\ 0, & n = 6, \ldots \end{cases}$$

$g[0] = 1,\ g[1] = 2,\ g[2] = 3,\ g[3] = 2,\ g[4] = 1$

(b) $h_1[n] * g[n] = h[n]$ ← total response

$\Rightarrow h_1[n]$ should have 3 non-zero samples $h_1[0],\ h_1[1],\ h_1[2]$

Using the expression $h[n] = \Sigma h_1[k]g[n-k]$ we find $h_1[0] = 1,\ h_1[1] = 2,\ h_1[2] = 1$

(c) We may use the relationship $\delta[n-k_1] * \delta[n-k_2] = \delta[n-k_1-k_2]$

$h[n] = \delta[n] + 4\delta[n-1] + 8\delta[n-2] + 10\delta[n-3] + 8\delta[n-4] + 4\delta[n-5] + \delta[n-6]$

$x[n] = \delta[n-1] - \delta[n-3]$

$y[n] = x[n] * h[n] = \delta[n-1] + 4\delta[n-2] + 8\delta[n-3] + 10\delta[n-4]$
$+ 8\delta[n-5] + 4\delta[n-6] + \delta[n-7] - \delta[n-3] - 4\delta[n-4] - 8\delta[n-5]$
$- 10\delta[n-6] - 8\delta[n-7] - 4\delta[n-8] - \delta[n-9] =$
$\delta[n-1] + 4\delta[n-2] + 7\delta[n-3] + 6\delta[n-4] - 6\delta[n-6] - 7\delta[n-7]$
$- 4\delta[n-8] - \delta[n-9]$

Problem 2

(a) $x(t) = \sum_{k=-\infty}^{k=+\infty} a_k e^{jk\omega_0 t} \Rightarrow$

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{k=+\infty} \left(a_k \, jk\omega_0\right) e^{jk\omega_0 t}$$

$\Rightarrow$ the function $\frac{dx(t)}{dt}$ has Fourier series $a_k \, jk\omega_0$

(b) (i) Real implies that $a_k = a_{-k}^*$. Since this is not true $x(t)$ is not real

(ii) Odd implies that $a_k = -a_{-k}$. Since this is not true $x(t)$ is not odd

(iii) The Fourier series of $\frac{dx(t)}{dt}$ are

$$b_k = \begin{cases} 0 & k = 0 \\ k\left(\frac{1}{3}\right)^{|k|} \omega_0 & \text{otherwise} \end{cases}$$

$$b_{-k} = -b_k \Rightarrow \frac{dx(t)}{dt} \text{ is odd}$$

(c) $y(t) = dx(t)/dt \Rightarrow Y(j\omega) = j\omega X(j\omega)$

(d) $x(t) = e^{-at} u(t)$

$$X(j\omega) = \int_0^{+\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a + j\omega}$$

(e) (i) $\left[(j\omega)^2 + 4j\omega + 3\right] Y(j\omega) = 2X(j\omega) \Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{-\omega^2 + 4j\omega + 3}$

$$\Rightarrow H(j\omega) = \frac{2}{(j\omega+1)(j\omega+3)} = \frac{(j\omega+3) - (j\omega+1)}{(j\omega+1)(j\omega+3)} =$$

$$= \frac{1}{j\omega+1} - \frac{1}{j\omega+3} \Rightarrow h(t) = e^{-t} u(t) - e^{-3t} u(t)$$

(ii) $Y(j\omega) = \dfrac{2}{(j\omega+1)(j\omega+3)^2}$

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Problem 3

(a) $$H(j\omega) = \frac{1}{(j\omega)^2 + 2j\omega + 1} = \frac{1}{(j\omega+1)^2}$$

$$H_1(j\omega) = \frac{1}{j\omega+1} \Rightarrow H(j\omega) = \frac{1}{H_1(j\omega)^2} \Rightarrow$$

$$|H(j\omega)| = \frac{1}{|H_1(j\omega)|^2} \Rightarrow 20\log|H(j\omega)| = 20\log|H_1(j\omega)|^2 =$$
$$= 2 \cdot 20\log|H_1(j\omega)|$$

$$\measuredangle H(j\omega) = 2 \measuredangle H_1(j\omega)$$

(b) $$X(j\omega) = \frac{1}{2+j\omega} \Rightarrow Y(j\omega) = \frac{1}{(j\omega+1)^2(2+j\omega)} = \frac{(2+j\omega)-(1+j\omega)}{(j\omega+1)^2(2+j\omega)} =$$

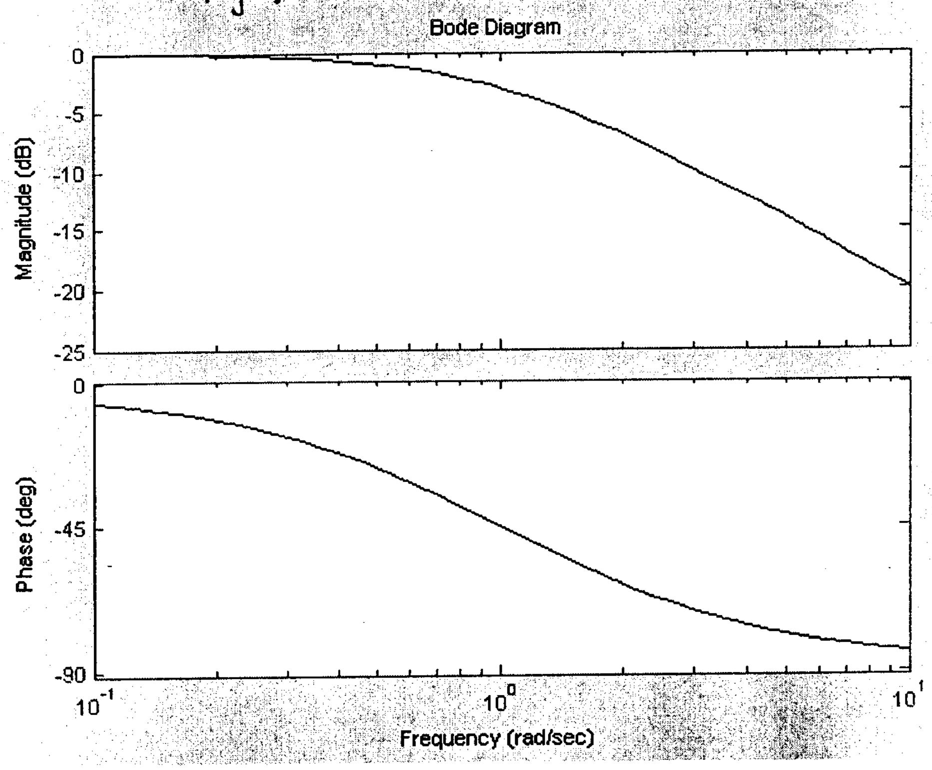
$$= \frac{1}{(j\omega+1)^2} - \frac{1}{(j\omega+1)(j\omega+2)} = \frac{1}{(j\omega+1)^2} - \frac{(j\omega+2)-(j\omega+1)}{(j\omega+1)(j\omega+2)} =$$

$$= \frac{1}{(j\omega+1)^2} - \frac{1}{j\omega+1} + \frac{1}{j\omega+2} \Rightarrow$$

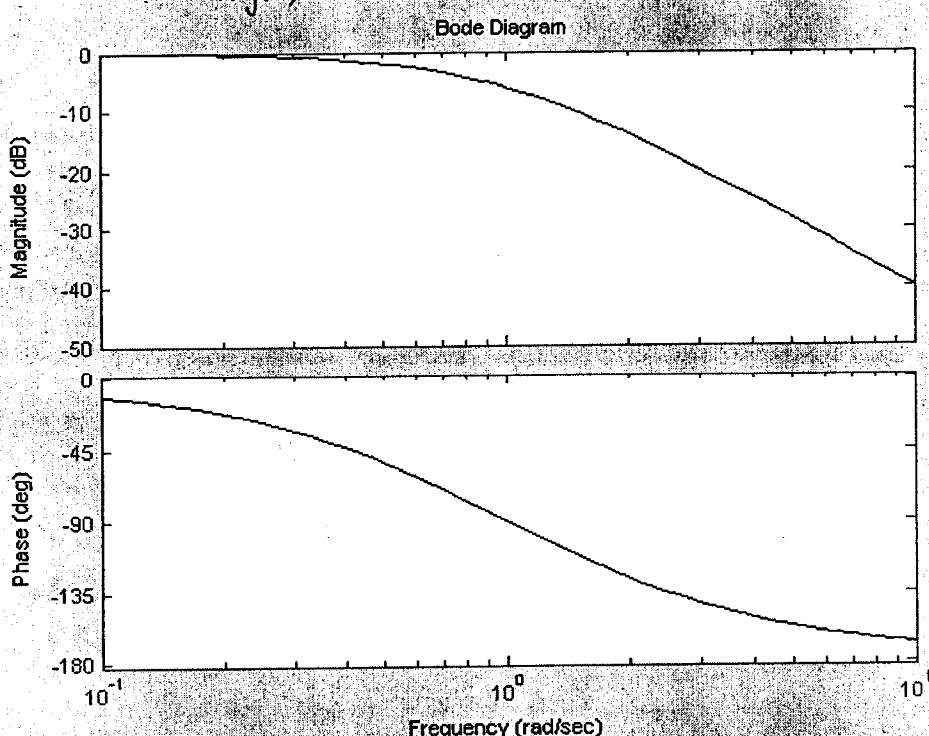
$$(t) = \left(te^{-t} - e^{-t}\right)u(t) + e^{-2t}u(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega) \Rightarrow 20\log Y(j\omega) = 20\log H(j\omega) + 20\log X(j\omega)$$
$$\measuredangle Y(j\omega) = \measuredangle H(j\omega) + \measuredangle X(j\omega)$$

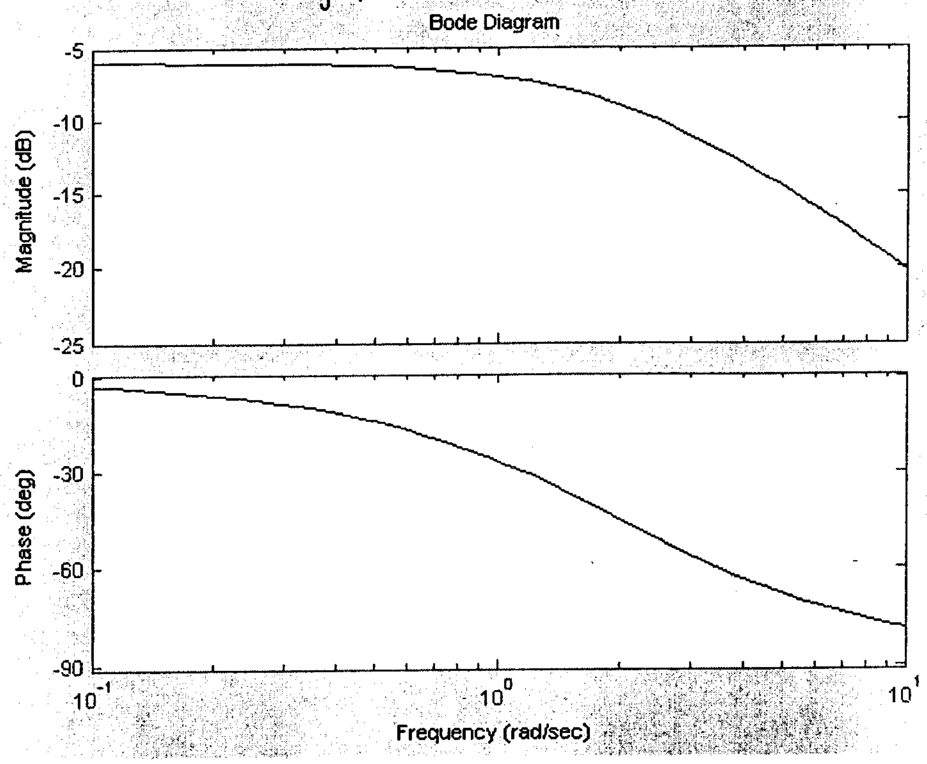
$H_1(j\omega)$



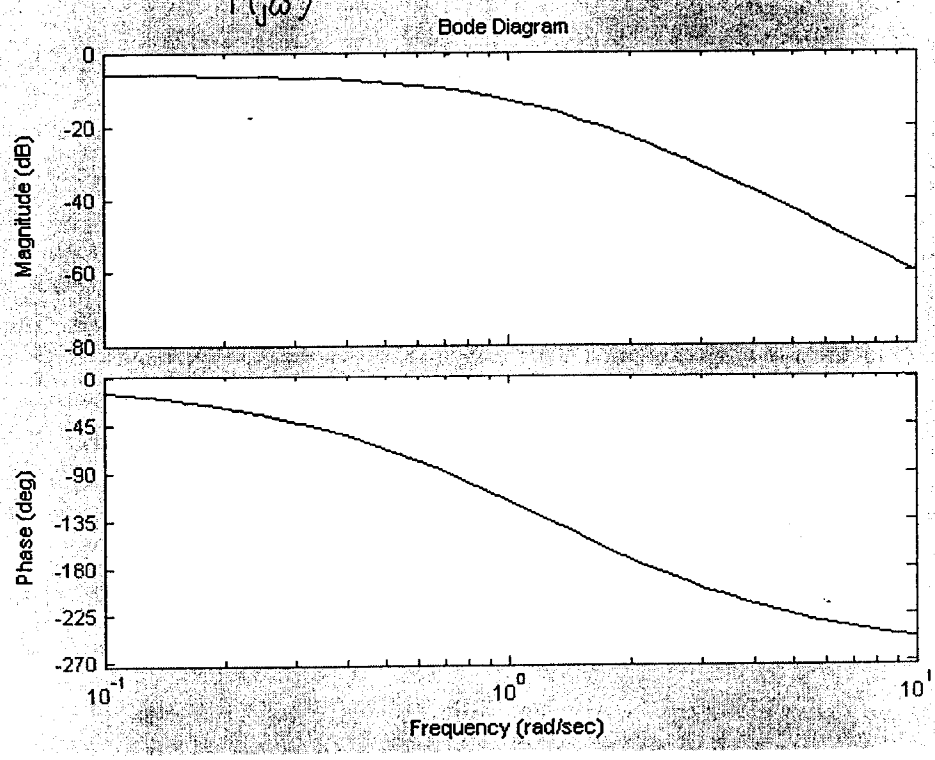
$H(j\omega)$



X(jω)


Bode Diagram
Magnitude (dB)
-5
-10
-15
-20
-25
Phase (deg)
0
-30
-60
-90
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

Y(jω)


Bode Diagram
Magnitude (dB)
0
-20
-40
-60
-80
Phase (deg)
0
-45
-90
-135
-180
-225
-270
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

Problem 4

(a) (i) $X(s) = \int_0^{+\infty} e^{-t} e^{-st}\, dt = \dfrac{1}{s+1}$

$H(s) = \dfrac{1}{s+3}$

(ii) $Y(s) = H(s)\,X(s) = \dfrac{1}{(s+3)(s+1)}$

(iii) $Y(s) = \dfrac{0.5[(s+3)-(s+1)]}{(s+3)(s+1)} \Rightarrow y(t) = 0.5\,e^{-t}\,u(t) - 0.5\,e^{-3t}\,u(t)$

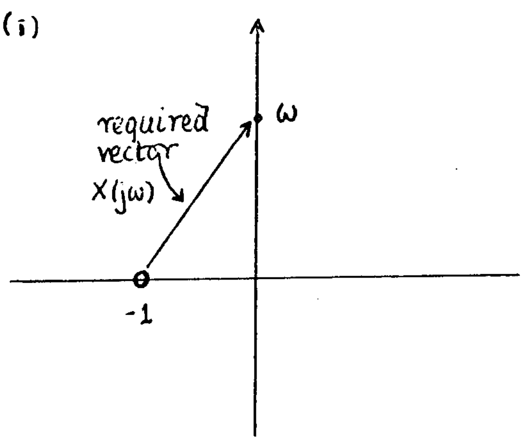
(iv) $y(t) = \int_?^? x(\tau)h(t-\tau)\,d\tau = \int_?^? e^{-\tau} e^{-3(t-\tau)}\, d\tau$

$$\left.\begin{array}{l} 0 \le \tau \le +\infty \\ 0 \le t-\tau \le +\infty \Rightarrow -\infty \le \tau - t \le 0 \Rightarrow -\infty \le \tau \le t \end{array}\right\} t \ge 0$$

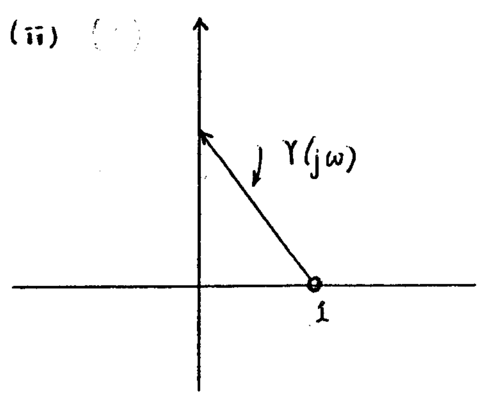
$y(t) = \int_0^t e^{-3t} e^{2\tau}\, d\tau = e^{-3t}\, 0.5\, e^{2\tau}\Big]_0^t = e^{-3t}\, 0.5\,(e^{2t} - 1) \Rightarrow$

$y(t) = 0.5e^{-t} - 0.5e^{-3t}, \quad t \ge 0 \Rightarrow y(t) = 0.5e^{-t}u(t) - 0.5e^{-3t}u(t)$

(b) (i)



(ii)



(iii) $|X(j\omega)| = |Y(j\omega)|$

$\measuredangle X(j\omega) = \pi - \measuredangle Y(j\omega)$

Problem 5

(a) (i) $X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{z}{z-a} \quad |z|>|a|$

(ii) $X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{n=0}^{+\infty} (a^{-1} z)^n = \frac{z}{z-a}, \quad |z|<|a|$

(iii) no, since two functions may have the same z transforms but different ROC's as in (i),(ii)

(b) $y[n] = x[n-m]$

$$Y(z) = x(0) z^{-m} + x(1) z^{-(m+1)} + \dots = z^{-m} X(z)$$

(c) $H(z) = \frac{1}{1 - \frac{5}{4} z^{-1} + \frac{3}{8} z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}$

(i) CAUSAL SYSTEM

$A(1-\frac{3}{4}z^{-1}) + B(1-\frac{1}{2}z^{-1}) = 1 \Rightarrow A+B=1$

$-\frac{3A}{4} - \frac{2B}{4} = 0 \Rightarrow -3A-2B=0 \Rightarrow -3A-2(1-A)=0 \Rightarrow$

$-3A-2+2A=0 \Rightarrow A=-2, \quad B=3$

$$H(z) = \underbrace{\frac{-2}{1-\frac{1}{2}z^{-1}}}_{\{ROC: |z|>\frac{1}{2}\}} + \underbrace{\frac{3}{1-\frac{3}{4}z^{-1}}}_{\{ROC: |z|>\frac{3}{4}\}}, \quad ROC: |z|>\frac{3}{4}$$

$$h(t) = \left[-2\left(\tfrac{1}{2}\right)^n + 3\left(\tfrac{3}{4}\right)^n\right] u[n]$$

(ii) $H(z)$ as in (i) ROC: $|z|<\frac{1}{2}$

$$h(t) = \left[2\left(\tfrac{1}{2}\right)^n - 3\left(\tfrac{3}{4}\right)^n\right] u[-n-1]$$