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IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2006

EEE/ISE PART II: MEng, BEng and ACGI

COMMUNICATIONS 2

Wednesday, 24 May 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : M. Petrou, M. Petrou

Second Marker(s) : J.A. Barria, J.A. Barria

1. **[Compulsory]**

a) Consider the signal $x(t) = m(t)\cos(\omega_c t + \theta)$ where $m(t)$ is some function of time, ω_c is a constant and θ is a random variable taking values from a uniform distribution in the range $[0, \pi]$.

i) Compute the ensemble mean value of $x(t)$. [6]

ii) Compute the ensemble mean value of $x(t)^2$. [6]

b) You receive a signal $n(t)$ which is known to be white Gaussian noise.

i) Explain what the term "white" means in the above expression. [2]

ii) Explain what the term "Gaussian" means in the above expression.

[2]

c) Starting from the representation of the noise as $n(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$ where A_k, f_k and θ_k are random variables, show that $n(t)$ may be written as

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

where $n_c(t)$ and $n_s(t)$ are suitably defined expressions (which you should derive) and $\omega_c \equiv 2\pi f_c$ with f_c being the carrier frequency of the modulation system used. [10]

d) You use noise signal $n(t)$ as an input to the system of Figure 1.1.

i) Show that the two branches of the system will output the two components $n_c(t)$ and $n_s(t)$. [10]

ii) Explain how you would choose the low pass filter in each branch so that the above is true. [4]

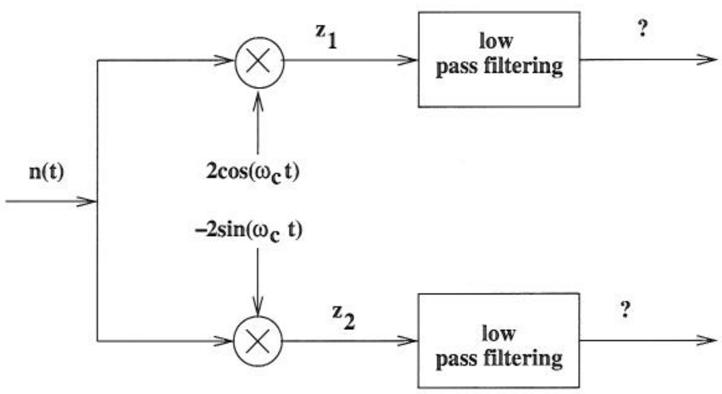

Figure 1.1 The system used in part (d) of Question 1

2. a) Explain what is meant by the term "ergodicity". [2]

b) When is a signal considered to be ergodic with respect to the mean? [3]

c) When is a signal considered to be ergodic with respect to the autocorrelation function? [4]

d) What use do we make of the assumption of ergodicity in signal processing? [3]

e) A 5-sample long digital signal is transmitted down a noisy communication line five times. Five versions of that signal are received, and they constitute an ensemble of signals. These versions are:

$$(2,3,4,6,5)$$
$$(3,2,6,6,3)$$
$$(5,5,3,2,5)$$
$$(1,3,5,5,6)$$
$$(9,7,2,1,1)$$

i) Calculate the ensemble mean signal. [3]

ii) Is the signal stationary with respect to the mean? [1]

iii) Is this signal ergodic with respect to the mean? [3]

iv) Calculate the ensemble autocorrelation function of this signal $R(t_1, t_3)$, where t_1 and t_3 refer to the times the first and the third samples of the signal are received. [5]

v) Calculate the temporal autocorrelation function of the second instantiation of this signal for shift $\tau = 2$. [5]

vi) Is this signal ergodic with respect to the autocorrelation function? [1]

3. Consider a message signal with a bandwidth of 10 kHz and an average power of $P = 20\ Watt$. Assume that the transmission channel does not attenuate the signal, but simply adds noise with a power spectral density of

$$S(f) = \begin{cases} N_0 & \text{for } |f| \leq 100 \times 10^3 Hz \\ N_0 \left(1 - \frac{|f| - 100 \times 10^3}{200 \times 10^3}\right) & \text{for } 100 \times 10^3 Hz \leq |f| \leq 200 \times 10^3 Hz \\ 0 & \text{elsewhere} \end{cases}$$

where $N_0 = 10^{-5}\ Watt/Hz$. At the output of the receiver a suitable filter is used to limit out-of-band noise.

a) What does "suitable filter" mean in the above statement? [3]

b) Plot the power spectral density of the noise signal. [3]

c) What is the predetection SNR, in dB, at the receiver if the transmission is base-band (no modulation scheme is used)? [9]

d) What is the predetection SNR at the receiver if DSB-SC modulation is used with carrier frequency of 150 kHz and amplitude $A_c = 1\ V$? [15]

4. a) Define the term "channel capacity". [3]

b) State the Hartley Shannon theorem. [3]

c) According to information theory, which property would make a modulation scheme ideal? [2]

d) Use the Hartley Shannon theorem to work out the relationship between the input and the output signal to noise ratio for the ideal modulation scheme. [7]

e) A source produces symbols from a four-symbol alphabet $\{A, B, C, D\}$, with corresponding frequencies $0.6, 0.04, 0.3, 0.06$, respectively.

i) Compute the entropy of this source. [3]

ii) What is the meaning of the term "source entropy"? [2]

iii) Use Huffman coding to construct a coding scheme for this source. [7]

iv) Compute the average codeword length for the coding scheme you constructed. [3]

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E2.4
Communications 2

SOLUTIONS TO EXAM QUESTIONS 2006

1. a) **Consider the signal $x(t) = m(t)\cos(\omega_c t + \theta)$ where $m(t)$ is some function of time, ω_c is a constant and θ is a random variable taking values from a uniform distribution in the range $[0, \pi]$.**

i) **Compute the ensemble mean value of $x(t)$.**

The only random variable is θ. It has prob. density function:

$$p(\theta) = \begin{cases} \frac{1}{\pi} & \text{for } 0 \leq \theta < \pi \\ 0 & \text{elsewhere} \end{cases} \tag{1.1}$$

The ensemble mean then is

$$\begin{aligned} < x(t) > &= \int_{-\infty}^{\infty} x(t)p(\theta)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} m(t)\cos(\omega_c t + \theta)d\theta \\ &= \frac{m(t)}{\pi}[\sin(\omega_c t + \theta)|_0^{\pi} \\ &= \frac{m(t)}{\pi}[\sin(\omega_c t + \pi) - \sin(\omega_c t)] \\ &= \frac{m(t)}{\pi}[-\sin(\omega_c t) - \sin(\omega_c t)] \\ &= -2\frac{m(t)}{\pi}\sin(\omega_c t) \end{aligned}$$

[*Calculation of new example*] [6]

ii) **Compute the ensemble mean value of $x(t)^2$.**

$$\begin{aligned} < x(t)^2 > &= \int_{-\infty}^{\infty} x(t)^2 p(\theta)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} m(t)^2\cos^2(\omega_c t + \theta)d\theta \\ &= \frac{m(t)^2}{\pi}\int_0^{\pi} \frac{1 + \cos(2\omega_c t + 2\theta)}{2}d\theta \\ &= \frac{m(t)^2}{2\pi}\left[\theta + \frac{\sin(2\omega_c t + 2\theta)}{2}\right|_0^{\pi} \\ &= \frac{m(t)^2}{2\pi}\pi \\ &= \frac{m(t)^2}{2} \end{aligned}$$

[*Calculation of new example*] [6]

b) **You receive a signal $n(t)$ which is known to be white Gaussian noise.**

i) **Explain what the term "white" means in the above expression.**

It means that the power spectral density of the signal is flat, ie it has the same value for all frequencies. [*Bookwork*] [2]

ii) **Explain what the term "Gaussian" means in the above expression.**

It means that the random number that affects the value of a signal afflicted by this noise is drawn from a Gaussian probability density function. [*Bookwork*] [2]

c) **Starting from the representation of the noise as $n(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$ where A_k, f_k and θ_k are random variables, show that $n(t)$ may be written as**

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t) \qquad (1.2)$$

where $n_c(t)$ and $n_s(t)$ are suitably defined expressions (which you should derive) and $\omega_c \equiv 2\pi f_c$ with f_c being the carrier frequency of the modulation system used.

We start by writing first $f_k = f_k - f_c + f_c$ and substituting into the given expression for $n(t)$:

$$\begin{aligned}
n(t) &= \sum_k A_k \cos(2\pi f_k t + \theta_k) \\
&= \sum_k A_k \cos\left(2\pi(f_k - f_c + f_c)t + \theta_k\right) \\
&= \sum_k A_k \cos\left(2\pi(f_k - f_c) + \theta_k + 2\pi f_c t\right) \\
&= \sum_k A_k \cos\left([2\pi(f_k - f_c) + \theta_k] + [2\pi f_c t]\right) \\
&= \sum_k A_k \left(\cos[2\pi(f_k - f_c) + \theta_k]\cos[2\pi f_c t] - \sin[2\pi(f_k - f_c) + \theta_k]\sin[2\pi f_c t]\right) \\
&= \cos[2\pi f_c t]\sum_k A_k \cos[2\pi(f_k - f_c) + \theta_k] - \sin[2\pi f_c t]\sum_k A_k \sin[2\pi(f_k - f_c) + \theta_k] \\
&= n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)
\end{aligned}$$

where

$$n_c(t) \equiv \sum_k A_k \cos[2\pi(f_k - f_c) + \theta_k] \qquad n_s(t) \equiv \sum_k A_k \sin[2\pi(f_k - f_c) + \theta_k] \qquad (1.3)$$

[*Bookwork*] [10]

d) **You use noise signal $n(t)$ as an input to the system of figure 1.1.**

i) **Show that the two branches of the system will output the two components $n_c(t)$ and $n_s(t)$.**

Write down expressions for the two signals z_1 and z_2:

$$\begin{aligned}
z_1 &= n(t)2\cos(\omega_c t) \\
&= (n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t))2\cos(\omega_c t) \\
&= n_c(t)2\cos^2(\omega_c t) - n_s(t)2\sin(\omega_c t)\cos(\omega_c t) \\
&= n_c(t)(1 + \cos(2\omega_c t)) - n_s(t)\sin(2\omega_c t)
\end{aligned}$$

LPF can be used to allow only $n_c(t)$ to pass.

$$\begin{aligned}
z_2 &= -n(t)2\sin(\omega_c t) \\
&= -(n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t))2\sin(\omega_c t) \\
&= -n_c(t)2\cos(\omega_c t)\sin(\omega_c t) + n_s(t)2\sin^2(\omega_c t) \\
&= -n_c(t)\sin(2\omega_c t) + n_s(t)(1 - \cos(2\omega_c t))
\end{aligned}$$

LPF can be used to allow only $n_s(t)$ to pass. [*Material related to a tutorial problem*] [10]

ii) **Explain how you would choose the low pass filter in each branch so that the above is true.**

The LPF should be chosen so that its bandwidth is less than $2f_c$ (where $\omega_c = 2\pi f_c$) to cut off the trigonometric terms in the output signals. Also, it should have flat response for frequencies below $2f_c$ so that it does not distort the component that passes. [*Material related to a tutorial problem*] [4]

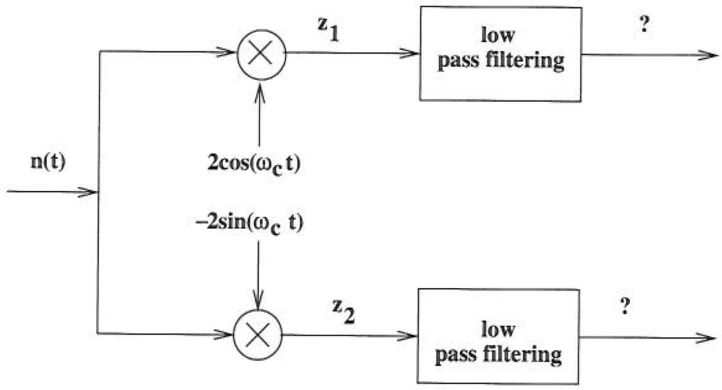

Figure 1.1 The system used in part (d) of question 1

2. a) **Explain what is meant by the term "ergodicity".**

The ensemble statistics of a random process are equal to its temporal statistics.

[Bookwork] [2]

b) **When is a signal considered to be ergodic with respect to the mean?**

When it is stationary with respect to the mean (ie it has the same ensemble mean at any instant in time) and its ensemble average is the same as the temporal average of any instantiation of it. *[Bookwork]* [3]

c) **When is a signal considered to be ergodic with respect to the autocorrelation function?**

When it is stationary with respect to its autocorrelation function (ie the ensemble autocorrelation function has the same value for any pair of instances at the same relative time shift from each other) and this value is the same as the value of its temporal autocorrelation function for the same time shift, and this is valid for all time shifts. *[Bookwork]* [4]

d) **What use do we make of the assumption of ergodicity in signal processing?**

We replace the calculation of ensemble statistics with the calculation of temporal statistics over the single instantiation we have at our disposal. *[Bookwork]* [3]

e) **A 5-sample long digital signal is transmitted down a noisy communication line five times. Five versions of that signal are received, and they constitute an ensemble of signals. These versions are:**

$$(2,3,4,6,5)$$
$$(3,2,6,6,3)$$
$$(5,5,3,2,5)$$
$$(1,3,5,5,6)$$
$$(9,7,2,1,1)$$

i) **Calculate the ensemble mean signal.**

Average the columns to get: $(4,4,4,4,4)$ *[New example]* [3]

ii) **Is the signal stationary with respect to the mean?**

Yes, because at any instant it has the same mean. *[New example]* [1]

iii) **Is this signal ergodic with respect to the mean?**

We must compute the average of each row. All rows give 4. So, the temporal average of any instantiation of the signal is the same as its ensemble average at any instant in time. So, it is ergodic with respect to the mean. *[New example]* [3]

iv) **Calculate the ensemble autocorrelation function of this signal $R(t_1,t_3)$, where t_1 and t_3 refer to the times the first and the third sample of the signal are received.**

$$R(t_1,t_3) = \frac{2\times 4+3\times 6+5\times 3+1\times 5+9\times 2}{5} = \frac{64}{5} = 12.8 \quad (2.1)$$

[New example] [5]

v) **Calculate the temporal autocorrelation function of the second instantiation of this signal for shift $\tau = 2$.**

$$R(2) = \frac{3 \times 6 + 2 \times 6 + 6 \times 3}{3} = \frac{48}{3} = 16 \tag{2.2}$$

[*New example*] [5]

vi) **Is this signal ergodic with respect to the autocorrelation function?**

No, because the temporal autocorrelation function of the second instantiation is different from the ensemble autocorrelation function for the same shift. [*New example*] [1]

3. **Consider a message signal with a bandwidth of $10kHz$ and an average power of $P = 20watt$. Assume that the transmission channel does not attenuate the signal, but simply adds noise with a power spectral density of**

$$S(f) = \begin{cases} N_0 & \text{for } |f| \leq 100 \times 10^3 Hz \\ N_0\left(1 - \frac{|f| - 100 \times 10^3}{200 \times 10^3}\right) & \text{for } 100 \times 10^3 Hz \leq |f| \leq 200 \times 10^3 Hz \\ 0 & \text{elsewhere} \end{cases} \quad (3.1)$$

where $N_0 = 10^{-5} watt/Hz$. At the output of the receiver a suitable filter is used to limit out-of-band noise.

a) **What does "suitable filter" mean in the above statement?**

It means that the filter will have the same bandwidth as the message, ie $10kHz$.

[New example] [3]

b) **Plot the power spectral density of the noise signal.**

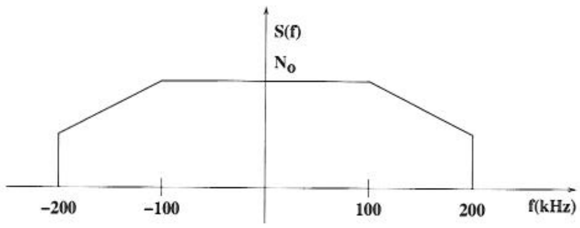

Figure 3.1 Power spectral density of noise

[New example] [3]

c) **What is the predetection SNR, in dB, at the receiver if the transmission is baseband (no modulation scheme is used)?**

The baseband transmission will only allow frequencies in the range $[-10, 10]$ kHz. So, the average power of the noise will be $P_N = 2N_0 10 \times 10^3 = 2 \times 10^{-5} \times 10^4 = 0.2watt$. Then

$$SNR = \frac{P}{P_N} = \frac{20}{0.2} = 100 \Rightarrow SNR = 10\log_{10} 100 = 20dB \quad (3.2)$$

[New example] [9]

d) **What is the predetection SNR at the receiver if DSB-SC modulation is used with carrier frequency of $150kHz$ and amplitude $A_c = 1v$?**

We must calculate the average power of the noise. The band of allowed frequencies will be $f_c + 10\ kHz$, ie from $140kHz$ to $160kHz$. At this range, the power spectral density of the noise is given by $S(f) = N_0\left(1 - \frac{|f| - 100 \times 10^3}{200 \times 10^3}\right)$. So

$$\begin{aligned} P_N &= 2\int_{140kHz}^{160kHz} S(f)df \\ &= 2\int_{140kHz}^{160kHz} N_0\left(1 - \frac{f - 100 \times 10^3}{200 \times 10^3}\right)df \\ &= 2\left\{N_0(160kHz - 140kHz) - N_0\frac{1}{200 \times 10^3}\left[\frac{f^2}{2} - 100 \times 10^3 f\right|_{140kHz}^{160kHz}\right\} \\ &= 2\left\{N_0 20kHz - N_0\frac{1}{200 \times 10^3}\left(\frac{160^2 \times 10^6 - 140^2 \times 10^6}{2} - 100 \times 10^3 \times 20 \times 10^3\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= 2\left\{N_0 20\times 10^3 - N_0\frac{10^3}{200}\left(\frac{160^2-140^2}{2}-2000\right)\right\}\\
&= 2\left\{N_0 20\times 10^3 - N_0\frac{10^3}{200}(3000-2000)\right\}\\
&= 2\left\{N_0 20\times 10^3 - N_0\times 10^3\times 5\right\}\\
&= 2\left\{N_0\times 15\times 10^3\right\}\\
&= 2\left\{10^{-5}\times 15\times 10^3\right\}\\
&= 0.30
\end{aligned}
$$

For DSB-SC the transmitted power is $P_T = \frac{A_c^2 P}{2} = \frac{P}{2} = 10 watt$. So,

$$SNR = 10\log_{10}\frac{10}{0.30} = 15.23 \tag{3.3}$$

[*New example*] [15]

4. a) **Define the term "channel capacity".**

It is the maximum rate at which error free information may be transmitted through the channel, even when the channel is noisy.

[*Bookwork*] [3]

b) **State the Hartley Shannon theorem.**

For additive white Gaussian noise, the capacity C of a channel is given by

$$C = B\log_2\left(1+\frac{S}{N}\right) \tag{4.1}$$

where B is the bandwidth of the channel, S is the average signal power at the receiver and N is the average noise power at the receiver.

[*Bookwork*] [3]

c) **According to information theory which property would make a modulation scheme ideal?**

The channel capacity should be the same before and after demodulation.

[*Bookwork*] [2]

d) **Use the Hartley Shannon theorem to work out the relationship between the input and the output signal to noise ratio for the ideal modulation scheme.**

The maximum rate at which information may arrive at the receiver is

$$C_{in} = B\log_2(1+SNR_{in}) \tag{4.2}$$

where SNR_{in} is the predetection signal to noise ratio at the input of the demodulator.

After demodulation, the signal is low pass filtered with bandwidth W, that of the message. The maximum rate at which information can leave the receiver is:

$$C_o = W\log_2(1+SNR_o) \tag{4.3}$$

where SNR_o is the SNR at the output of the post-detection filter.

For an ideal demodulation scheme, $C_{in} = C_o$. Therefore:

$$B\log_2(1+SNR_{in}) = W\log_2(1+SNR_o) \Rightarrow SNR_o = [1+SNR_{in}]^{B/W} - 1 \tag{4.4}$$

[*Bookwork*] [7]

e) **A source produces symbols from a four-symbol alphabet $\{A,B,C,D\}$, with corresponding frequencies $0.6, 0.04, 0.3, 0.06$, respectively.**

i) **Compute the entropy of this source.**

$$\begin{aligned}
H &= -0.6\times\log_2 0.6 - 0.04\log_2 0.04 - 0.3\log_2 0.3 - 0.06\log_2 0.06 \\
&= -3.32(0.6\times\log_{10}0.6 + 0.04\log_{10}0.04 + 0.3\log_{10}0.3 + 0.06\log_{10}0.06 \\
&= 3.32(0.133 + 0.056 + 0.157 + 0.073) \\
&= 1.391
\end{aligned}$$

[*New example*] [3]

ii) **What is the meaning of the term "source entropy"?**

It gives the average amount of information per source symbol. [*Book-work*] [2]

iii) **Use Huffman coding to construct a coding scheme for this source.**

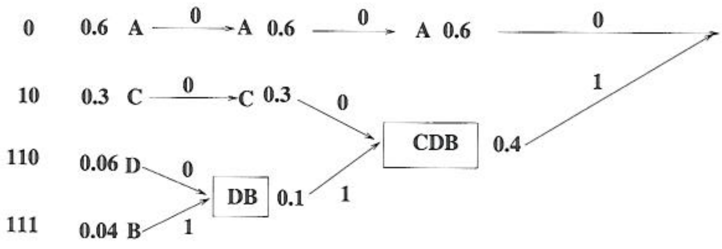

Figure 4.1 Huffman coding of the 4 symbol source

[*New example*] [7]

iv) **Compute the average codeword length for the coding scheme you constructed.**

$$\bar{L} = 0.6 \times 1 + 0.3 \times 2 + 0.06 \times 3 + 0.04 \times 3 = 1.5 \qquad (4.5)$$

[*New example*] [3]