7.

(a) Assume a separated harmonic solution in the form: $y(x, t) = X(x) \exp(j\omega_0 t)$

Substitute into the wave equation to get: $d^4X/dx^4 = k^4X$

Where: $k^4 = \omega_0^2\rho A/EI$

Substitution shows that $X(x) = A \sin(kx) + B \cos(kx) + C \sinh(kx) + D \cosh(kx)$ is a solution to the ODE.

[4]

(b) For a beam built in at one end and free at the other, the boundary conditions are:

No deflection or slope at the LH end, so y = 0 (1), dy/dx = 0 (2) at x = 0

No moment or shear force at the RH end so $d^2y/dx^2 = 0$ (3); $d^3y/dx^3 = 0$ (4) at x = L

[2]

Differentiating:

$X'(x) = Ak \cos(kx) - Bk \sin(kx) + Ck \cosh(kx) + Dk \sinh(kx)$

$X''(x) = -Ak^2 \sin(kx) - Bk^2 \cos(kx) + Ck^2 \sinh(kx) + Dk^2 \cosh(kx)$

$X'''(x) = -Ak^3 \cos(kx) + Bk^3 \sin(kx) + Ck^3 \cosh(kx) + Dk^3 \sinh(kx)$

Applying the boundary conditions:

BC1: B + D = 0 so D = -B (5)

BC2: A + C = 0 so C = -A (6)

BC3: $-A \sin(kL) - B \cos(kL) + C \sinh(kL) + D \cosh(kL) = 0$ (7)

BC4: $-A \cos(kL) + B \sin(kL) + C \cosh(kL) + D \sinh(kL) = 0$ (8)

Substituting equations 5 and 6 into equations 7 and 8, we get:

$A \{\sin(kL) + \sinh(kL)\} + B \{\cos(kL) + \cosh(kL)\} = 0$ (9)

$A \{\cos(kL) + \cosh(kL)\} - B \{\sin(kL) - \sinh(kL)\} = 0$ (10)

Combining equations (9) and (10) to eliminate B, we get:

$A\{2 + 2 \cos(kL) \cosh(kL)\} = 0$ (11)

Equation (11) can then be written as:

$\cos(\beta) \cosh(\beta) = -1$ where $\beta = kL$ (12)

Now, $k^4 = \omega_0^2\rho A/EI$, so $\omega_0 = k^2 \sqrt{EI/\rho A} = (\beta/L)^2 \sqrt{EI/\rho A}$ (13)

[8]

(c) If cos(β) cosh (β) = -1, then numerical solution gives:

b	**cos(b)**	**cosh (b)**	**cos(b) cosh (b)**
2.000	-0.4161	3.762	-1.565
1.800	-0.2272	3.107	-0.706
1.900	-0.3233	3.418	-1.105
1.880	-0.3043	3.353	-1.020
1.870	-0.2947	3.321	-0.979
1.875	-0.2995	3.337	-0.999

Hence, to reasonable accuracy, $\beta = 1.875$ and $\beta^2 = 3.52$

[3]

For the silicon beam:

$A = 2 \times 100 \times 10^{-12} = 2 \times 10^{-10}\ m^2$

$I = bd^3/12 = 100 \times 2^3 \times 10^{-24} = 8 \times 10^{-22}\ m^4$

$E = 1.08 \times 10^{11}\ N/m^2$

$\rho = 2330\ kg/m^3$

$\sqrt{EI/\rho A} = 0.013616\ m^2/s$

$\omega_0 = (\beta/L)^2 \sqrt{EI/\rho A} = (3.52/10^{-6}) \times 0.013616 = 47{,}928\ rad/s$

$f_0 = \omega_0/2\pi = 7{,}628\ kHz$

[3]

8.

(a) *Gravitational force* depends on volume, and hence scales as $O[L^3]$.

Elastic force might be exemplified by the force provided by a cantilever spring. In this case, $F = kx$, where x is the end displacement, the stiffness is $k = 3EI/L^3$, E is Young's modulus, and $I = bd^3/12$ is the second moment of a rectangular beam of breadth b and depth d. Combining, we obtain $F = Exbd^3/4L^3$, which scales as $O[L^2]$.

Surface tension acts on the perimeter of a liquid surface, and hence scales as $O[L]$.

[3]

<u>Gravitational force ($O[L^3]$) versus elastic force ($O[L^2]$)</u> Gravity must eventually overcome elastic force as size increases. There is therefore an upper limit to the size of self-supporting structures. In contrast, gravitational force becomes relatively insignificant compared to elastic force as size reduces, e.g. to the microstructure size domain.

[2]

<u>Gravitational force ($O[L^3]$) versus surface tension force ($O[L]$)</u> Surface tension must eventually overcome gravitational force as size reduces. An application lies in surface tension driven self-assembly of microstructures.

[2]

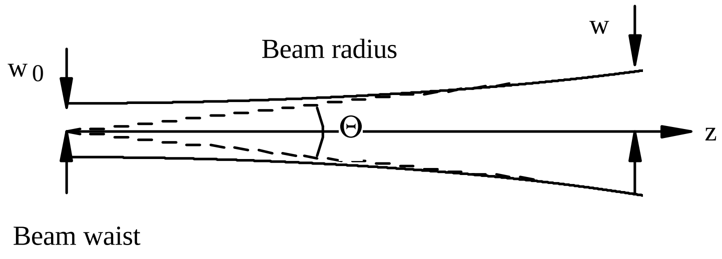
<u>Elastic force ($O[L^2]$) versus surface tension force ($O[L]$)</u> Surface tension must eventually overcome elastic force as size reduces. One important consequence is the collapse of suspended microstructures in the drying step that follows sacrificial layer etching.

[2]

(b) The expressions for beam radius and phase-front curvature have simple approximations in the near- and far-field regimes:

$(w/w_0)^2 = \{1 + (z/z_0)^2\}$ reduces to $w/w_0 \approx 1$ for $z << z_0$ and $w/w_0 \approx z/z_0$ for $z >> z_0$

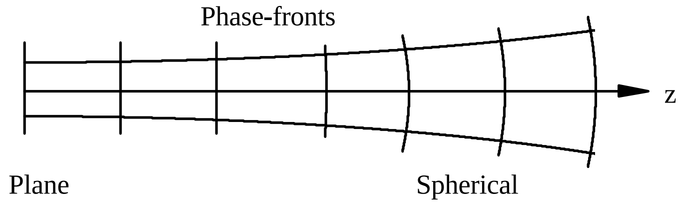
The beam radius is therefore roughly constant in the near field, increasing linearly in the far field.



[3]

$R = z\{1 + (z_0/z)^2\}$ reduces to $R \approx \infty$ for $z << z_0$ and $R \approx z$ for $z >> z_0$

Phase-fronts are therefore flat in the near field, and spherically curved in the far-field, with a centre at $z = 0$.



[3]

The distance z_0 represents the transition between the near and far-field regimes

[1]

The radius of a Gaussian varies as $w^2 = w_0^2\{1 + (z/z_0)^2\}$

Since $z_0 = k_0 w_0^2/2$, we can write $w^2 = w_0^2 + 4z^2/(k_0^2 w_0^2)$.

Differentiating with wrt w_0, we get $2w\, dw/dw_0|_{z = const} = 2w_0 - 8z^2/(k_0^2 w_0^3)$.

At the minimum, the RHS = 0, so that $w_0 = \sqrt{\{2z/k_0\}} = \sqrt{\{\lambda z/\pi\}}$.

[2]

The optimum beam waist radius therefore scales as $O[L^{0.5}]$.

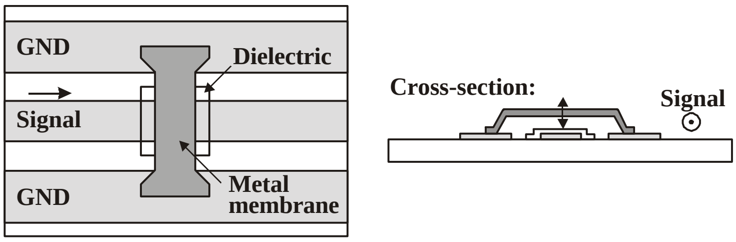
[2]

9.

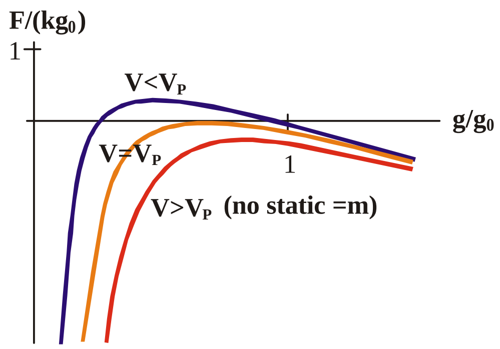
(a) Advantages: low insertion loss; good isolation; low power consumption; good linearity
Limitations: slow switching speed; high actuation voltage

[6]

(b)



Combination of parallel plate actuator and linear spring gives non-linear force-displacement curve with single maximum:

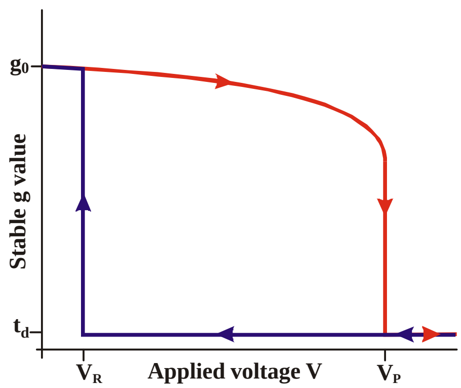


Below critical voltage Vp, there is a stable equilibrium where F = 0, dF/dg < 0.

For V > Vp, there is no equilibrium point and the bridge snaps down.

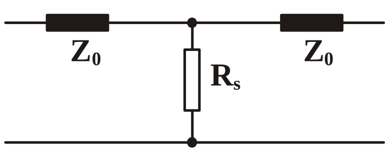
Once snap-down has occurred, V must be reduced until F $\to$ 0 at minimum gap.

Resulting variation of bridge height g with applied voltage exhibits hysteresis:



[10]

(c) Need to calculate $|S_{21}|^2$ for case where transmission line has a resistive shunt Rs:



$S_{21} = 2(R_s//Z_0)/[(R_s//Z_0) + Z_0]$
$\approx 2R_s/Z_0$ if $R_s << Z_0$

and so $|S_{21}|^2 \approx 4R_s^2/Z_0^2$ as required.

[4]