MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

E1

PAPER 3

QUESTION

SOLUTION 4

Please write on this side only, legibly and neatly, between the margins

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ie $\frac{d}{dt}\underline{x} = A\,\underline{x}$ 2

Eigenvalues $|A-\lambda I| = 0$ ie $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$ 2

ie $\lambda = 1 \pm 2 = 3$ or -1

Eigenvector corr. to $\lambda = -1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 2

Eigenvector corr. to $\lambda = 3$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(No need to normalise).

General solution $\underline{x} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}$ 3

evalues

Initial conds. $\Rightarrow$ $\left.\begin{aligned} 1 &= a+b \\ 0 &= -a+b \end{aligned}\right\} \Rightarrow b = \frac{1}{2}, a = \frac{1}{2}$ 3

Thus $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^{-t}+e^{3t}) \\ \frac{1}{2}(-e^{-t}+e^{3t}) \end{pmatrix} \sim \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 3

as $t \to \infty$

15

Setter : R.L. JACOBS Setter's signature : R.L. Jacobs

Checker : J.D. GIBBON Checker's signature : J.D. Gibbon

E2

(i) $$W = 1/(z-1)^2 = \frac{[(x-1)-iy]^2}{[(x-1)^2+y^2]^2} = u+iv$$

$$u = \frac{(x-1)^2 - y^2}{[(x-1)^2+y^2]^2} \qquad v = \frac{-2y(x-1)}{[\quad]^2}$$

$$\therefore\ u^2+v^2 = \frac{1}{[(x-1)^2+y^2]^2} = \frac{1}{[R^2]^2} = 1/R^4$$ Circle in the w-plane radius R^{-2}. 5

(ii) $$w = 1/z = \frac{x-iy}{x^2+y^2}$$

$$\therefore\ u = \frac{x}{x^2+y^2}\ ;\ v = \frac{-y}{x^2+y^2}.$$

Also $$z = 1/w \Rightarrow x = \frac{u}{u^2+v^2}\ ;\ y = -\frac{v}{u^2+v^2}.$$

$$\therefore\ r^2 = (x-1)^2+(y-1)^2 = \frac{(u-u^2-v^2)^2 + (v+u^2+v^2)^2}{(u^2+v^2)^2}$$

$$\therefore\ \underline{r^2(u^2+v^2)^2 = (u^2+v^2)[1-2u+2v] + 2(u^2+v^2)^2}$$

a) Hence if $r^2 = 2$; $1 = 2u - 2v \Rightarrow v = u - \frac{1}{2}$ Straight Line. 4

b) However if $r^2 = 1$; $(u^2+v^2)(1-2u+2v) + (u^2+v^2)^2 = 0$

$$\therefore\ 1 - 2u + 2v + u^2 + v^2 = 0$$

$$\therefore\ (u-1)^2 + (v+1)^2 = 1$$ Circle, radius 1 centred at $(1,-1)$. 6

Note: The student may attempt the question the opposite way.
e.g. $u - \frac{1}{2} = v \Rightarrow \frac{x}{x^2+y^2} - \frac{1}{2} = \frac{-y}{x^2+y^2} \rightarrow (x-1)^2+(y-1)^2 = 2$ etc.
Either way is acceptable.

(Seen similar)

Setter: J.D. Gibbon Checker: R.L.U.

E3

For $\dfrac{e^{iz}}{z(z^2+1)(z^2+4)} = \dfrac{e^{iz}}{z(z+i)(z-i)(z+2i)(z-2i)}$

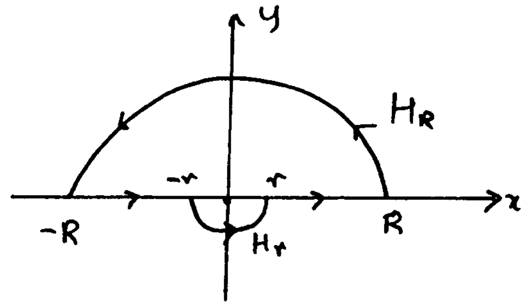
Poles at $z=0$, $z=\pm i$, $z=\pm 2i$ 1

1) Residue at $z=0$ is $1/4$.

2) " " $z=i$ is $\dfrac{e^{-1}}{i\times 2i\times(3i)\times(-3i)} = -e^{-1}/6$. 2

3) " " $z=2i$ is $\dfrac{e^{-2}}{2i\times(-3)\times(4i)} = e^{-2}/24$. 2

$\lim_{r\to 0}\int_{H_r}\dfrac{e^{iz}\,dz}{z(z^2+1)(z^2+4)}$, $z=re^{i\theta}$



$= \lim_{r\to 0}\int_{H_r}\dfrac{e^{ire^{i\theta}}\,\cancel{ire^{i\theta}}\,d\theta}{\cancel{re^{i\theta}}(r^2e^{2i\theta}+1)(r^2e^{2i\theta}+4)}$

$= \dfrac{i}{4}\int_{\pi}^{2\pi}d\theta = \pi i/4$ 2

H_R: the circle $z=Re^{i\theta}$, $\theta: 0\to\pi$

H_r: the circle $z=re^{i\theta}$, $\theta: \pi\to 2\pi$

Over the full contour C (picture)

$\oint_C \dfrac{e^{iz}\,dz}{z(z^2+1)(z^2+4)} = 2\pi i\times\{\text{Sum of Residues at } z=0,\ i,\ 2i\}$

$= 2\pi i\times\left\{\frac{1}{4}-\frac{1}{6e}+\frac{1}{24e^2}\right\}$ 4

$= \lim_{\substack{R\to\infty\\ r\to 0}}\left\{\int_{-R}^{-r}+\int_{r}^{R}\right\}\dfrac{e^{ix}\,dx}{x(x^2+1)(x^2+4)} + \lim_{r\to 0}\int_{H_r} + \lim_{R\to\infty}\int_{H_R}\dfrac{e^{iz}\,dz}{z(\ )(\ )}$

Zero from Jordan's Lemma: a) $m=1$ b) $F(z)\to 0$ as $R\to\infty$ c) Only sings are poles. 3

Together we have

$\int_{-\infty}^{\infty}\dfrac{e^{ix}\,dx}{x(x^2+1)(x^2+4)} + \dfrac{\pi i}{4} = 2\pi i\left\{\frac{1}{4}-\frac{1}{6e}+\frac{1}{24e^2}\right\}$

The cosine part is zero, leaving.

$\int_{-\infty}^{\infty}\dfrac{\sin x\,dx}{x(x^2+1)(x^2+4)} = \pi\left\{\frac{1}{4}-\frac{1}{3e}+\frac{1}{12e^2}\right\} = \dfrac{\pi(3e^2-4e+1)}{12e^2}$ 1

$= \pi\,\dfrac{(3e-1)(e-1)}{12e^2}$

(Seen similar)

Setter: J.D. Gibbon Checker [signature]

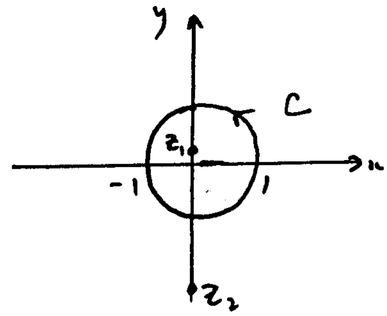
E4 JDG

$z = e^{i\theta}$ $\quad dz = iz\,d\theta$ $\quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$= \frac{1}{2i}(z - 1/z)$ 4

$$\therefore \frac{d\theta}{2+\sin\theta} = \frac{dz}{iz\left[2 + \frac{1}{2i}(z - \frac{1}{z})\right]}$$

$$= \frac{2\,dz}{z^2 + 4iz - 1}$$



$$\therefore I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2\oint_C \frac{dz}{z^2+4iz-1}$$ 4

$z^2 + 4iz - 1 = (z - z_1)(z - z_2)$ $\quad z_1 = (-2+\sqrt{3})i$ In C

$\therefore$ Two simple poles $\longrightarrow$ $\begin{cases} z_2 = (-2-\sqrt{3})i & \text{outside C} \\ z_2 \text{ does not contribute} \end{cases}$ 3

Residue at z_1 $= \dfrac{1}{z_1 - z_2} = \dfrac{1}{2i\sqrt{3}}$. 2

By the Residue Theorem

$$I = 2\oint_C \frac{dz}{z^2+4iz-1} = 4\pi i \times \text{Residue at the pole } z = z_1$$ 2

$$= \frac{2\pi}{\sqrt{3}}$$

(Seen similar)

Setter: J.D. Gibbon Checker: Dr C Hobbs

E5 JDG

$$\bar f(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \qquad \bar f^{*}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t'} f^{*}(t')\,dt'$$

$$\therefore \int_{-\infty}^{\infty} \bar f(\omega)\bar f^{*}(\omega)\,d\omega = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt\right)\left(\int_{-\infty}^{\infty} e^{i\omega t} f^{*}(t')\,dt'\right)d\omega$$

$$= \int_{-\infty}^{\infty}\left(\iint_{-\infty}^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} e^{i\omega(t'-t)}\,d\omega}_{2\pi\delta(t'-t)}\Big] f(t)\, f^{*}(t')\,dt'\right)d\omega$$

$$= 2\pi\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t) f^{*}(t')\,\delta(t'-t)\,dt'\right)dt$$

$$= 2\pi\int_{-\infty}^{\infty} |f(t)|^{2}\,dt$$

(Bookwork) ~~4~~ 5

(i) $$\bar\Lambda(\omega) = \int_{-1}^{0} e^{-i\omega t}(1+t)\,dt + \int_{0}^{1} e^{-i\omega t}(1-t)\,dt$$

$$= \frac{i}{\omega}\left(e^{-\omega} - e^{\omega}\right) + \int_{-1}^{0} t\,e^{-i\omega t}\,dt - \int_{0}^{1} t\,e^{-i\omega t}\,dt$$

Now $\int t\,e^{-i\omega t}\,dt = \frac{i}{\omega}\left[t e^{-i\omega t}\right] + \frac{1}{\omega^{2}}e^{-i\omega t}]$

$$\therefore \bar\Lambda(\omega) = \frac{i}{\omega}\cancel{(e^{-\omega i} - e^{i\omega})} + \frac{i}{\omega}\cancel{e^{i\omega}} + \frac{1}{\omega^{2}}\left[1 - e^{i\omega}\right] - \cancel{\frac{i}{\omega}e^{-i\omega}} - \frac{e^{-i\omega}}{\omega^{2}} + \frac{1}{\omega^{2}}$$

$$= \frac{1}{\omega^{2}}\left[2 - e^{i\omega} - e^{-i\omega}\right] = \frac{2}{\omega^{2}}\left[1 - \cos\omega\right] = \frac{4\sin^{2}\frac{1}{2}\omega}{\omega^{2}} = \mathrm{sinc}^{2}\omega$$ ~~4~~ 5

(ii) $$\int_{-\infty}^{\infty}\mathrm{sinc}^{4}\omega\,d\omega = \int_{-\infty}^{\infty}|\mathrm{sinc}^{2}\omega|^{2}\,d\omega = \int_{-\infty}^{\infty}|\bar f(\omega)|^{2}\,d\omega$$

where $\bar f(\omega) = \mathrm{sinc}^{2}\omega \Rightarrow f(t) = \Lambda(t)$

$\therefore$ By Parseval, $\int_{-\infty}^{\infty}\mathrm{sinc}^{4}\omega\,d\omega = 2\pi\int_{-\infty}^{\infty}|\Lambda(t)|^{2}\,dt$

$$= 2\pi\left\{\int_{-1}^{0}(1+t)^{2}\,dt + \int_{0}^{1}(1-t)^{2}\,dt\right\}$$

$$= 2\pi\left\{\left[t + t^{2} + \tfrac{1}{3}t^{3}\right]_{-1}^{0} + \left[t - t^{2} + \tfrac{1}{3}t^{3}\right]_{0}^{1}\right\}$$ ~~4~~ 5

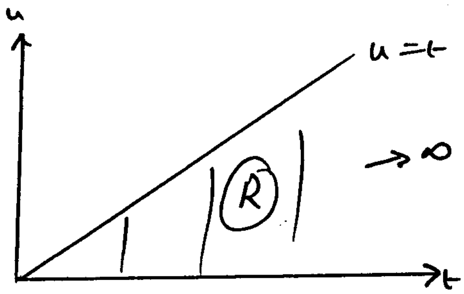
$$= 2\pi\left\{-\left[-1 + 1 - \tfrac{1}{3}\right] + \left[1 - 1 + \tfrac{1}{3}\right]\right\}$$

$$= 4\pi/3$$

Setter: J.D. Gibbon Checker: HERBERT

JDG

$$\mathcal{L}(f*g) = \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt$$



Exchanging order of integration to t first & u second, we have 2

$$\mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_u^\infty e^{-st}g(t-u)\,dt\right)du$$

Write $\tau = t-u$ so limits in τ are $0 \to \infty$

$$\mathcal{L}(f*g) = \int_0^\infty f(u)\left(\int_0^\infty e^{-s(\tau+u)}g(\tau)\,d\tau\right)du$$ 3

$$= \int_0^\infty e^{-su}f(u)\left(\int_0^\infty e^{-s\tau}g(\tau)\,d\tau\right)du = \bar{f}(s)\bar{g}(s)$$

Bookwork

$$\frac{s}{(s^2+1)^2} = \frac{s}{s^2+1}\cdot\frac{1}{s^2+1} = \bar{f}(s)\,\bar{g}(s)$$

$$\bar{f}(s) = \frac{s}{s^2+1} \Rightarrow f(t) = \cos t \;;\; \bar{g}(s) = \frac{1}{s^2+1} \Rightarrow g(t) = \sin t$$ 5

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \int_0^t \cos u \sin(t-u)\,du$$

$\sin 2u = 2\sin u\cos u$

$\cos 2u = 2\cos^2 u - 1$

$= 1 - 2\sin^2 u$

$$= \sin t\int_0^t \cos^2 u\,du - \cos t\int_0^t \cos u\sin u\,du$$

$$= \tfrac{1}{2}\sin t\int_0^t (1+\cos 2u)\,du - \tfrac{1}{2}\cos t\int_0^t \sin 2u\,du$$

$$= \tfrac{1}{2}\sin t\left[u + \tfrac{1}{2}\sin 2u\right]_0^t + \tfrac{1}{4}\cos t\left[\cos 2u\right]_0^t$$ 5

$$= \tfrac{1}{2}\sin t\left[t + \tfrac{1}{2}\sin 2t\right] + \tfrac{1}{4}\cos t\left[\cos 2t - 1\right]$$

$$= \tfrac{1}{2}\left\{\sin t\left[t + \sin t\cos t - \cos t\sin t\right]\right\}$$

$$= \tfrac{1}{2}t\sin t$$

Setter: J.D. Gibbon Checker: [signature]

$$\ddot{x} + 2\dot{x} + 5x = f(t) \qquad x(0) = \dot{x}(0) = 0$$

$$\mathcal{L}(\dot{x}) = s\bar{x}(s) - x(0) = s\bar{x}(s)$$

$$\mathcal{L}(\ddot{x}) = s^2\bar{x}(s) - s\,x(0) - \dot{x}(0) = s^2\bar{x}(s)$$

$$\therefore \quad (s^2 + 2s + 5)\,\bar{x}(s) = \bar{f}(s)$$ 3

$$\therefore \quad \bar{x}(s) = \bar{f}(s)\,\bar{g}(s)$$

where $\bar{g}(s) = \dfrac{1}{s^2 + 2s + 5} = \dfrac{1}{(s+1)^2 + 4}$ 4

$$= \frac{1}{2} \cdot \frac{2}{(s+1)^2 + 2^2}$$

We know that $\mathcal{L}(\sin 2t) = \dfrac{2}{s^2 + 2^2}$

therefore, by the shift Theorem 4

$$g(t) = \frac{1}{2} \cdot e^{-t} \sin 2t$$

Hence

$$x(t) = \mathcal{L}^{-1}\left[\bar{f}(s)\bar{g}(s)\right]$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du$$ 4

$$\therefore \quad x(t) = \frac{1}{2}\int_0^t e^{-u} \sin 2u\, f(t-u)\, du.$$

Seen similar

L8

PAPER II (3)
QUESTION
SOLUTION 23

(i) $\underline{F} = (x+3y^2)\hat{i} + (y-2z)\hat{j} + (x+\alpha z)\hat{k}$

(a) $\mathrm{div}\,\underline{F} = \frac{\partial}{\partial x}(x+3y^2) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\alpha z)$

$= 1+1+\alpha = \underline{\underline{2+\alpha}}$ 2

(b) $\mathrm{Curl}\,\underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+3y^2 & y-2z & x+\alpha z \end{vmatrix}$ 5

$= \hat{i}(0-(-2)) - \hat{j}(1-0) + \hat{k}(0-6y)$ 2

$= \underline{\underline{2\hat{i}-\hat{j}-6y\hat{k}}}$

(c) $\mathrm{div}(\mathrm{curl}\,\underline{F}) = 0$ (as is always the case) 1

(ii) $\underline{v} = \underline{\omega}\times\underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$ letting $\underline{\omega} = \omega_1\hat{i}+\omega_2\hat{j}+\omega_3\hat{k}$

$= \hat{i}(\omega_2 z-\omega_3 y) - \hat{j}(\omega_1 z-\omega_3 x) + \hat{k}(\omega_1 y-\omega_2 x)$ 2

$\therefore \mathrm{Curl}\,\underline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x \end{vmatrix} = \hat{i}(\omega_1-(-\omega_1)) - \hat{j}(-\omega_2-\omega_2) + \hat{k}(\omega_3-(-\omega_3))$ 5

$= 2\underline{\omega}$

$\therefore \underline{\underline{\underline{\omega} = \frac{1}{2}\mathrm{curl}\,\underline{v}}}$, as required. 3

(iii) $\mathrm{Curl}(f(r)\underline{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xf & yf & zf \end{vmatrix}$

$= \hat{i}\left(zf'\frac{\partial r}{\partial y} - yf'\frac{\partial r}{\partial z}\right) - \hat{j}\left(zf'\frac{\partial r}{\partial x} - xf'\frac{\partial r}{\partial z}\right) + \hat{k}\left(yf'\frac{\partial r}{\partial x} - xf'\frac{\partial r}{\partial y}\right)$ 2

(Since $r^2 = x^2+y^2+z^2$ we have $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$) 2

$= \hat{i}\left(\cancel{\frac{yz}{r}f'} - \cancel{\frac{yz}{r}f'}\right) - \hat{j}\left(\cancel{\frac{xz}{r}f'} - \cancel{\frac{xz}{r}f'}\right) + \hat{k}\left(\cancel{\frac{xy}{r}f'} - \cancel{\frac{xy}{r}f'}\right) = \underline{\underline{0}}$ 1

Total 15

Setter : A. WALTON　　Setter's signature : Andrew Walton
Checker : A. GOGOLIN　　Checker's signature : A. Gogolin

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

E9

PAPER II(3)

QUESTION

SOLUTION 24

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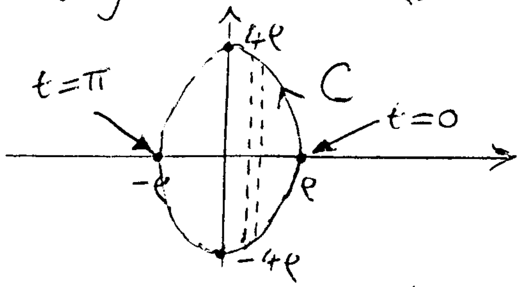
(i) $\left.\begin{array}{l} x = \rho\cos t \\ y = 4\rho\sin t \end{array}\right\} \rightarrow \dfrac{x^2}{\rho^2} + \dfrac{y^2}{(4\rho)^2} = 1$ ellipse.

3 for sketch



$dx = -\rho\sin t\,dt$

$dy = 4\rho\cos t\,dt$

3

$$\text{Area } A = \left|\int y\,dx\right| \quad \left(\text{or } \left|\int x\,dy\right|\right)$$

$$= \left|\int_0^{2\pi}(4\rho\sin t)(-\rho\sin t)\,dt\right|$$

$$\therefore\ A = 4\rho^2\int_0^{2\pi}\underbrace{\sin^2 t}_{\frac{1}{2}-\frac{1}{2}\cos 2t}\,dt = \underline{\underline{4\pi\rho^2}}$$

(Alternatively, use Jacobian or quote formula $A = \pi ab$ with $a = \rho$, $b = 4\rho$)

2

(ii) $\operatorname{div}\underline{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y}$ with $F_1 = 2x + 3y$, $F_2 = x + y$

$= 2 + 1 = \underline{\underline{3}}$

5

(iii) $Q = \int_C F_1\,dy - F_2\,dx = \int_0^{2\pi}(2\rho\cos t + 12\rho\sin t)(4\rho\cos t)\,dt$

$- \int_0^{2\pi}(\rho\cos t + 4\rho\sin t)(-\rho\sin t)\,dt$

Now, $\int_0^{2\pi}\sin t\cos t\,dt = 0$

$\therefore\ Q = \int_0^{2\pi}(8\rho^2\cos^2 t + 4\rho^2\sin^2 t)\,dt$ $\left(\& \int_0^{2\pi}\cos^2 t\,dt = \int_0^{2\pi}\sin^2 t\,dt = \pi\right)$

$= \underline{\underline{12\pi\rho^2}}$

2

(iv) Using answers to (i), (ii) & (iii) we see that

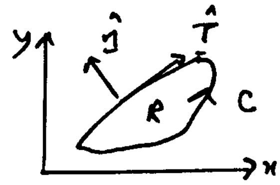
$$\underline{\underline{\operatorname{div}\underline{F} = \frac{Q}{A} = 3}}.$$

Total $\boxed{15}$

Setter : A. WALTON Setter's signature : Andrew Walton

Checker : A. GOGOLIN Checker's signature : A. Gogolin

G.T. $\oint_C (P\,dx + Q\,dy) = \iint_R (Q_x - P_y)\,dx\,dy$



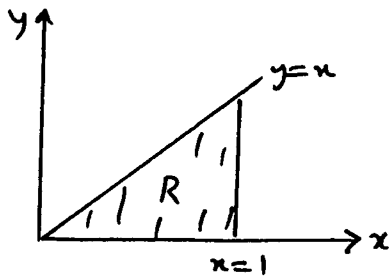
$\hat{\underline{T}} = d\underline{r}/ds = \hat{\underline{i}}\frac{dx}{ds} + \hat{\underline{j}}\frac{dy}{ds}$ $\quad \hat{\underline{n}}\cdot\hat{\underline{T}} = 0$ 6

$\hat{\underline{n}} = \hat{\underline{i}}\frac{dy}{ds} - \hat{\underline{j}}\frac{dx}{ds}$ $\quad \underline{u} = \hat{\underline{i}}Q - \hat{\underline{j}}P$

$\therefore$ $\mathrm{div}\,\underline{u} = Q_x - P_y$ $\quad (\underline{u}\cdot\hat{\underline{n}})ds = P\,dx + Q\,dy \rightarrow$ Div. Thm.

$\underline{u} = \frac{x^2 y}{1+y^2}\hat{\underline{i}} + [x\ln(1+y^2)]\hat{\underline{j}} \Rightarrow \mathrm{div}\,\underline{u} = \frac{4xy}{1+y^2}$ 3

$\therefore \oint_C (\underline{u}\cdot\hat{\underline{n}})\,ds = 4\iint_R \frac{xy}{1+y^2}\,dx\,dy$



$= 4\int_0^1 x\left(\int_0^x \frac{y\,dy}{1+y^2}\right)dx$ 3

$= 2\int_0^1 x\,[\ln(1+y^2)]_0^x\,dx$

$= 2\int_0^1 x\ln(1+x^2)\,dx$

$= \int_1^2 \ln u\,du$ $\quad u = 1+x^2$ 3

$= [u(\ln u - 1)]_1^2$

$= 2(\ln 2 - 1) - (\ln 1 - 1)$

$= 2\ln 2 - 1.$

(Seen similar)

Signature: J.D. Gibbon Checker: [signature]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003-2004

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E 11

PAPER 3

EE2

QUESTION

33

SOLUTION

1.

(a) $P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$ 1

$P(A \mid B)/P(B \mid A) = (\frac{1}{4} \div \frac{1}{3})/(\frac{1}{4} \div \frac{1}{2}) = \frac{3}{2}$ 2

$P(\text{exactly one of } A, B) = P(A \cup B) - P(A \cap B) = \frac{7}{12} - \frac{1}{4} = \frac{1}{3}$ 2

(b) (i) prob $= \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$ 1

(ii) prob $= 2 \times \frac{2}{6} \times \frac{4}{5} = \frac{8}{15}$ 2

(iii) prob $= \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$ 2

(c) 3 compts: prob = P(110 or 011 or 111) (where 1=fail, 0=non-fail)

$= 2p^2(1-p) + p^3 = p^2(2-p)$ 2

4 compts: prob = P(11xx or 1011 or 011x or 0011) (where x=0 or 1)

$= p^2 + p^3(1-p) + p^2(1-p) + p^2(1-p)^2 = p^2(3-2p)$ 3

15

Setter: MJ CROWDER

Setter's signature: MJ Crowder

Checker: AT WALDEN

Checker's signature:

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003-2004

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E 12

PAPER 3

EE2

QUESTION

34

SOLUTION

2.

calculate k: $1 = \int f(x,y)dxdy = \int_0^1 dx \int_0^{x^2} dy\{kx(x-y)\} = k\int_0^1 dx[x^2y - \frac{1}{2}xy^2]_0^{x^2}$

$= k\int_0^1 (x^4 - \frac{1}{2}x^5)dx = k[\frac{1}{5}x^5 - \frac{1}{12}x^6]_0^1 = \frac{7}{60}k \quad \Rightarrow k = 60/7$ 3

marginals: $f_X(x) = \int_0^{x^2} kx(x-y)dy = \frac{60}{7}[x^2y - \frac{1}{2}xy^2]_0^{x^2} = \frac{60}{7}(x^4 - \frac{1}{2}x^5)$ on (0,1) 2

$f_Y(y) = \int_{\sqrt{y}}^1 kx(x-y)dx = \frac{60}{7}[\frac{1}{3}x^3 - \frac{1}{2}yx^2]_{\sqrt{y}}^1$ 2

$= \frac{60}{7}\{(\frac{1}{3} - \frac{1}{2}y) - (\frac{1}{3}y^{3/2} - \frac{1}{2}y^2)\}$ on (0,1)

criterion: '$f(x,y) = f_X(x)f_Y(y)$ for all x, y' is *not* satisfied 2

evaluate: $\mathrm{E}(X^2 - Y) = \mathrm{E}(X^2) - \mathrm{E}(Y) = \int_0^1 x^2 f_X(x)dx - \int_0^1 y f_Y(y)dy$

$= \frac{60}{7}[\frac{1}{7}x^7 - \frac{1}{16}x^8]_0^1 - \frac{60}{7}[(\frac{1}{6}(y^2 - y^3) - (\frac{2}{21}y^{7/2} - \frac{1}{8}y^4)]_0^1$ 2

$= \frac{60}{7}(\frac{9}{7\times16} - \frac{5}{21\times8}) = \frac{85}{196} = 0.4337$

$\mathrm{P}(Y < \frac{1}{2} \mid X < \frac{1}{2}) = 1$ since $Y < X^2$ 1

$\mathrm{P}(Y < \frac{1}{2} \mid X = 0.9) = \int_0^{\frac{1}{2}} f(y \mid x = 0.9)dy = \int_0^{\frac{1}{2}}\{f(0.9,y)/f_X(0.9)\}dy$

$= \int_0^{\frac{1}{2}} \frac{0.9k(0.9-y)}{k(0.9^4 - \frac{1}{2}0.9^5)}dy = (0.9^3 - \frac{1}{2}0.9^4)^{-1}[0.9y - \frac{1}{2}y^2]_0^{\frac{1}{2}}$ 3

$= 0.9^{-3}(1 - 0.45)^{-1}(0.45 - 0.125) = \frac{0.325}{0.729\times0.55} = 0.811$

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Setter: MJ CROWDER Setter's signature: MJ Crowder

Checker: AT WALDEN Checker's signature: