IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2007

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

None

Friday, 25 May 10:00 am

CORRECTED COPY

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti, P.L. Dragotti
Second Marker(s) : M.K. Gurcan, M.K. Gurcan

Special Information for the Invigilators: none

Information for Candidates

The trigonometric Fourier series of a periodic signal $x(t)$ of period $T_0 = 2\pi/\omega_0$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

with

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt, \quad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t dt, \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t dt.$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{with} \quad D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

Some Fourier Transforms

$$\cos\omega_0 t \iff \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$$

$$\text{rect}(\tfrac{t}{\tau}) \iff \tau\text{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\text{sinc}(Wt) \iff \text{rect}(\tfrac{\omega}{2W})$$

$$\tfrac{\alpha^2}{2\pi}\text{sinc}^2\left(\tfrac{\alpha t}{2}\right) \iff \Delta\left(\tfrac{\omega}{\alpha}\right)$$

Some useful trigonometric identities

$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y).$$

Euler' formula

$$e^{jx} = \cos x + j\sin x.$$

Frequency modulation by a sinusoidal signal

$$\varphi_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t.$$

where $\beta = \Delta f/B$.
Table of Bessel Function values (recall that $|J_n(\beta)| = |J_{-n}(\beta)|$):

n	$\beta = 1$	$\beta = 2$	$\beta = 5$	$\beta = 10$
0	0.765	0.224	-0.178	-0.246
1	0.440	0.577	-0.328	0.043
2	0.115	0.353	0.047	0.255

The Questions

1. This question is compulsory.

   (a) Consider the signal $x(t) = e^{-5t}u(t)$, where $u(t)$ is the unit step function defined by

   $$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

   i. Compute the energy of $x(t)$.

   [4]

   ii. Compute the Fourier transform of $x(t)$.

   [4]

   iii. Compute the energy of $x(t)$ using Parseval's theorem.

   [4]

   (b) The output signal from a full AM modulator is

   $$x(t) = 10\cos(8000t) + 20\cos(10000t) + 10\cos(12000t).$$

   i. Sketch and dimension the Fourier transform of $x(t)$.

   [4]

   ii. Determine the modulating signal $m(t)$ and the carrier $c(t)$.

   [4]

   iii. Compute the power efficiency η.

   [4]

(c) Consider the signal $x(t) = 10\cos 100t$.

i. Sketch and dimension the spectrum of the DSB-SC modulated signal $s(t) = 2x(t)\cos 1000t$.

[4]

ii. From the spectrum of $s(t)$, identify the upper sideband (USB) and the lower sideband (LSB) spectra.

[2]

iii. From the USB spectrum, write the exact expression of the USB modulated signal $\varphi_{USB}(t)$.

[2]

(d) Consider the FM signal

$$\varphi(t) = \cos[2\pi f_0 t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha]$$

where $m(t) = 10\cos 2\pi f_m t$ and $k_f = 200\pi$. Using Carson's rule, the bandwidth of $\varphi(t)$ is 4 kHz. Determine the frequency f_m of $m(t)$.

[4]

(e) A 50 Ω transmission line is connected to a 100 Ω line with a matched termination. A sine wave propagating in the former is incident on the junction. Find

i. The voltage reflection coefficient k_v.

[2]

ii. Show how a resistor R connected in parallel at the junction can eliminate this reflection and find its value.

[2]

2. Consider the FM signal

$$\varphi(t) = 10\cos[1000t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha]$$

where $m(t) = \cos 100t$.

(a) Assume that $k_f = 100$. Determine the modulation index β of the frequency modulated signal.

[7]

(b) Using Carson's rule, determine the bandwidth of the frequency modulated signal.

[7]

(c) Sketch the spectrum of the modulated signal (plot only the frequency components that lie within the bandwidth derived using Carson's rule).

[8]

(d) Consider now the phase modulated signal

$$\varphi(t) = 10\cos[1000t + k_p m(t)]$$

where $m(t) = \cos 100t$. Find the value of k_p that leads to the same instantaneous maximum frequency of the modulated signal of Part (a).

[8]

3. Consider the periodic signal $x(t)$ shown in Figure 1.

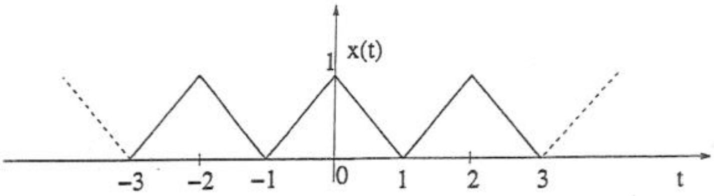


Figure 1: The periodic signal $x(t)$.

(a) Find the power of $x(t)$.

[6]

(b) Compute the coefficients a_0, a_n and b_n, of the trigonometric Fourier series of $x(t)$.

[6]

(c) Compute the coefficients D_n of the exponential Fourier series of $x(t)$.

[6]

(d) The signal $x(t)$ is fed to a filter $h(t)$ giving output $y(t)$. The frequency response of the filter is

$$H(\omega) = \begin{cases} 1 & \textit{for} \ |\omega| \leq \alpha \text{ rad/s} \\ 0 & \textit{otherwise} \end{cases}$$

where $\alpha = 3.1\pi$ rad/s. Compute the power of the output $y(t)$.

[6]

(e) Find the range of possible values of α that lead to $P_y = 0.75P_x$ where P_y and P_x are the power of $y(t)$ and $x(t)$ respectively.

[6]

4. Suppose the signal $x(t) = m(t) + \cos 1000t$ is applied to a non-linear system whose output is $y(t) = x(t) + \frac{1}{2}x^2(t)$ and assume $m(t) = \frac{1}{\pi}\text{sinc}(t)$.

(a) Sketch and dimension the Fourier transform of $m(t)$.

[6]

(b) Sketch and dimension the Fourier transform of $y(t)$.

[6]

(c) The signal $y(t)$ is fed to an ideal band-pass filter $h(t)$ giving output $z(t)$. The frequency response of the filter is

$$H(\omega) = \begin{cases} 1 & \text{for } \quad 950 \leq \omega \leq 1050 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$$

Write the exact expression of the output $z(t)$.

[6]

(d) Can you retrieve $m(t)$ from $z(t)$ using an envelop detector? Justify your answer.

[6]

(e) Now assume the message $m(t)$ has the Fourier transform shown in Figure 2. Find the exact expression of $m(t)$.

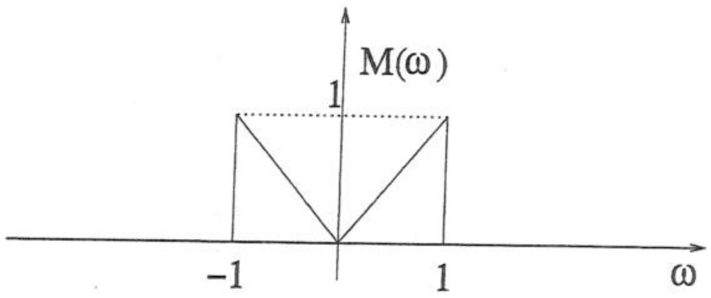


Figure 2: Fourier transform of $m(t)$.

[6]

QUESTION 1

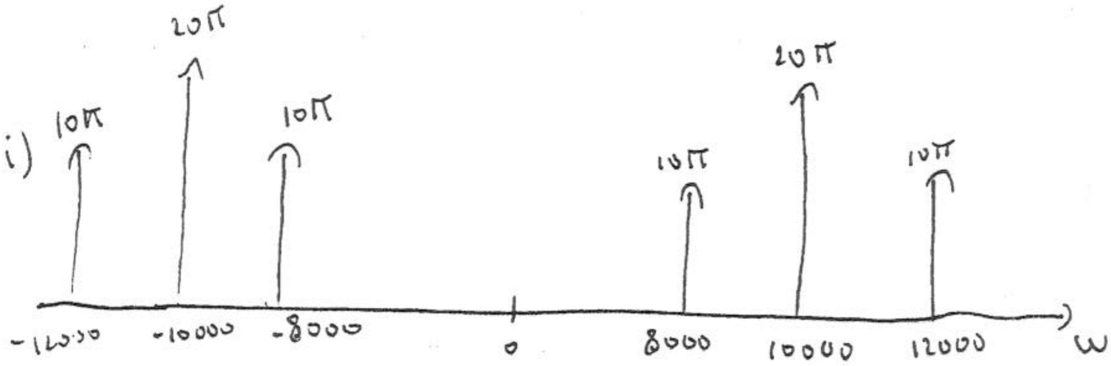
a)

i. $E_x = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{0}^{\infty} e^{-10t}\, dt = \frac{1}{10}$

ii. $X(w) = \int_{-\infty}^{\infty} e^{-5t} u(t)\, e^{-jwt}\, dt = \int_{0}^{\infty} e^{-(5+jw)t}\, dt = \frac{1}{5+jw}$

iii. $E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(w)|^2\, dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{25+w^2}\, dw = \frac{1}{10}$

b) i)



ii.

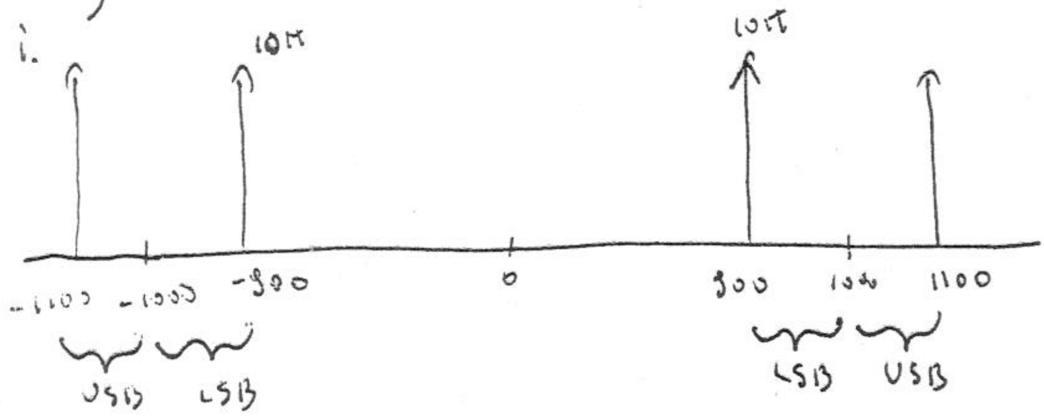
$m(t) = 20 \cos 2000t$

$c(t) = 20 \cos 10000t$

iii. $\eta = \frac{P_s}{P_s + P_c} = \frac{\frac{(20)^2}{4}}{\frac{(20)^2}{4} + \frac{(20)^2}{2}} = \frac{1}{3}$

c)

i.



iii $\varphi_{USB} = \cos 1100t$

d) $B_{FM} = 2\left(f_m + \frac{k_f m_p}{2\pi} \right) = 4000\ Hz$

$2\left(f_m + 1000 \right) = 4000\ Hz \Rightarrow f_m = 1000\ Hz$

(e) i. $K_V = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - 50}{150} = \frac{1}{3}$

ii. $\frac{R \cdot 100}{R + 100} = 50 \Rightarrow R = 100\ \Omega$

QUESTION 2

a) INSTANTANEOUS MAX FREQUENCIES

$$\omega_{MAX}^{PM} = \omega_c + k_p \dot{m}_p = \omega_c + 100\, k_p$$

$$\omega_{MAX}^{FM} = \omega_c + k_f m_p = \omega_c + k_f$$

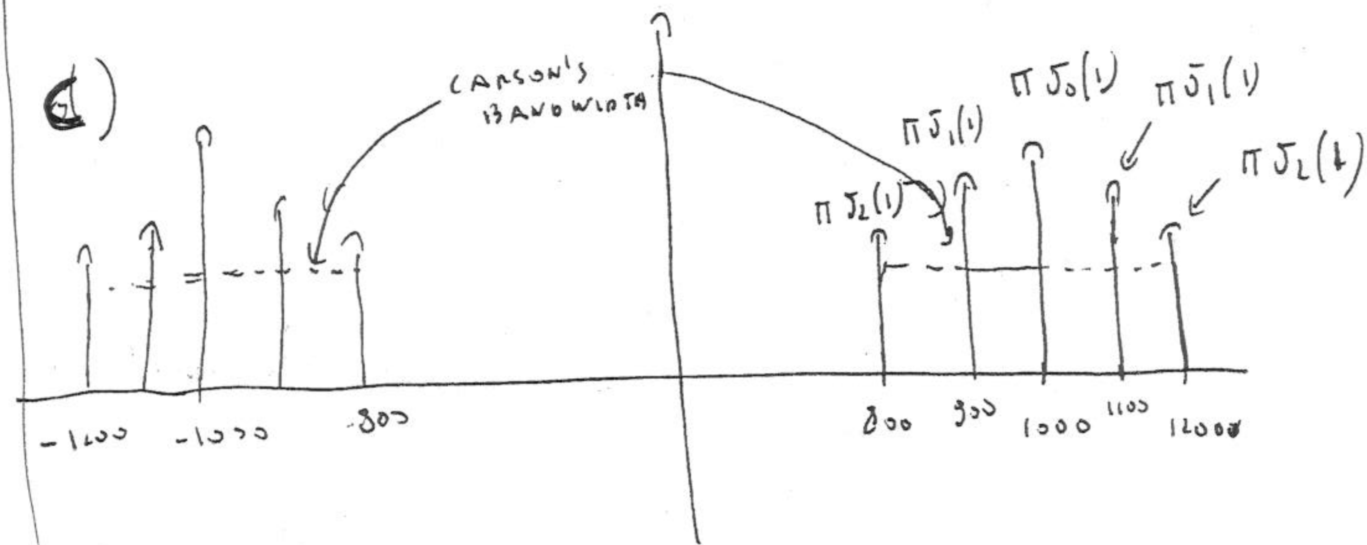
$\Rightarrow k_f = 100 k_p \Rightarrow$

$k_p = 1$,

a) $$\beta = \frac{\Delta f}{B} = \frac{k_f m_p}{2\pi \cdot B} = \frac{100 \cdot 1}{100} = 1$$

b) $$B = 2(\beta + 1) f_m = \frac{400}{2\pi}\ \text{Hz}$$

c)



QUESTION 3

a) $T_0 = 2 \qquad \omega_0 = \frac{2\pi}{T_0} = \pi$

$$P_x = \frac{1}{T_0}\int_{-1}^{1} x^2(t)\,dt = \frac{2}{T_0}\int_0^1 (1-t)^2 dt =$$

$$= \int_0^1 (1+t^2-2t)dt = 1 + \frac{1}{3} - 1 = \frac{1}{3}$$

b) $b_n = 0$ SINCE $x(t)$ IS EVEN

$$a_0 = \frac{1}{T_0}\int_{-1}^{1} x(t)\,dt = \int_0^1 (1-t)\,dt = 1 - \frac{1}{2} = \frac{1}{2}$$

$$a_n = \frac{2}{T_0}\int_{-1}^{1} x(t)\cos n\omega_0 t\,dt = \frac{4}{T_0}\int_0^1 (1-t)\cos n\omega_0 t\,dt$$

$$= 2\left[\int_0^1 \cos n\omega_0 t\,dt - \int_0^1 t\cos n\omega_0 t\,dt\right] =$$

$$= \frac{2}{\pi^2 n^2}\left(1 - \cos n\pi\right) = \frac{2}{\pi^2 n^2}\left(1-(-1)^n\right)$$

$$x(t) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(1-\cos n\pi)}{n^2}\cos n\pi t =$$

$$= \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos n\pi t.$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\left[\cos\pi t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t \ldots\right]$$

(c)

$$|D_n| = \frac{C_n}{2} = \frac{a_n}{2} \qquad D_0 = a_0$$

$$\angle D_n = 0$$

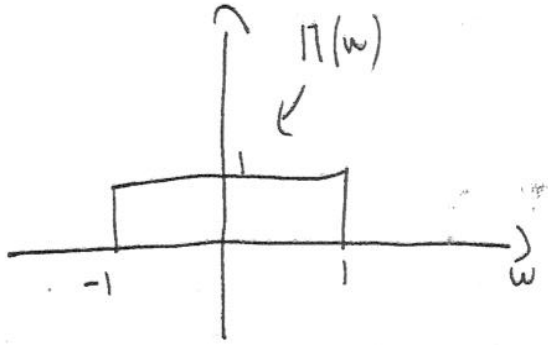
d) $\cancel{P_y = \frac{1}{4} + \frac{1}{2}\left(\frac{16}{\pi^4} + \frac{1}{\pi^2}\frac{16}{81\pi^4}\right) = \frac{1}{4} + \frac{20}{3\pi^4} = 0.318}$

$$P_y = \frac{1}{4} + \frac{1}{2}\left(\frac{16}{\pi^4} + \frac{16}{\pi^4 81}\right) = \frac{1}{4} + \frac{8}{\pi^4}\left(1 + \frac{1}{81}\right) \simeq 0.32$$

e) $0 < \lambda < \pi$

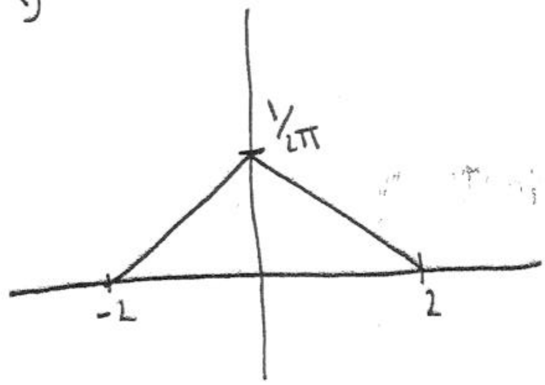
QUESTION 4

a) $M(\omega) = \text{RECT}\left(\frac{\omega}{2}\right)$

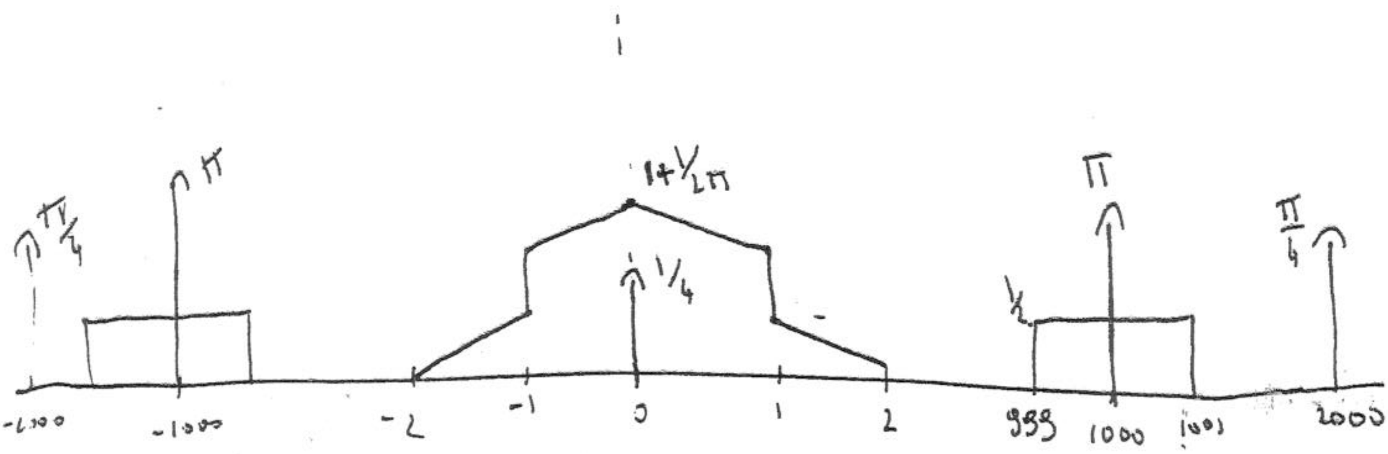


b) $y(t) = m(t) + \cos 1000t + \frac{1}{2} m^2(t) + \frac{1}{4} + \frac{1}{4} \cos 1000t +$
$m(t) \cos 1000t.$

$\frac{1}{2} m^2(t) \Leftrightarrow \frac{1}{4\pi} M(\omega) * M(\omega)$



$Y(\omega)$

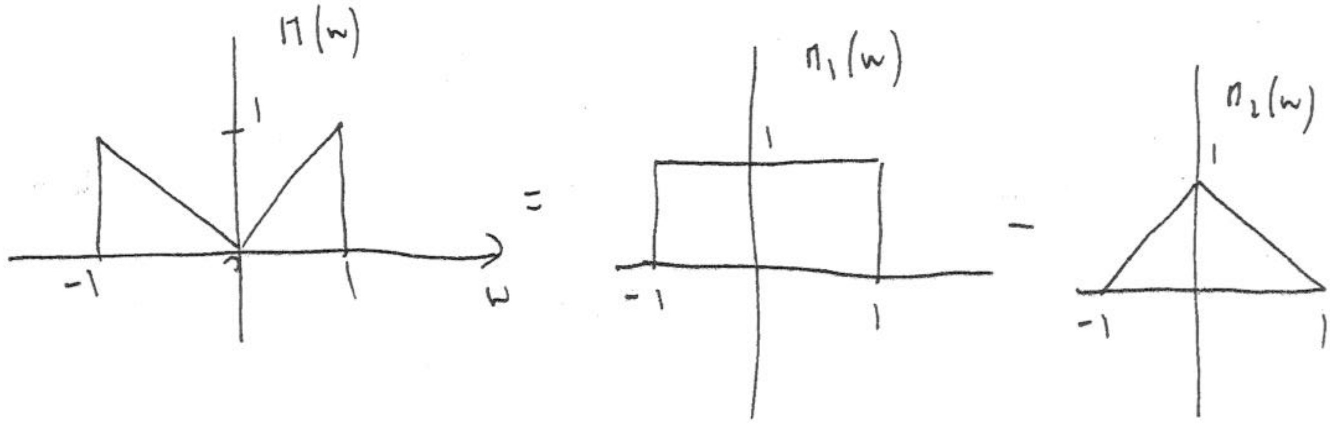


c)

$$\varphi(t) = (1 + m(t)) \cos 1000t$$

d) YES, SINCE $m_p = 1$ AND THEREFORE
$1 + m(t) \geq 0 \quad \forall t$

e)



$$M(w) = M_1(w) - M_2(w)$$

THUS

$$m(t) = m_1(t) - m_2(t) = \frac{1}{\pi} \operatorname{SINC} t - \frac{1}{2\pi} \operatorname{SINC}^2\left(\frac{t}{2}\right)$$