IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2004

EEE/ISE PART I: MEng, BEng and ACGI

COMMUNICATIONS 1

Friday, 28 May 10:00 am

Time allowed: 2:00 hours

There are FIVE questions on this paper.

Corrected Copy

Answer THREE questions.

All questions carry equal marks

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : P.L. Dragotti
Second Marker(s) : E.M. Yeatman

Special Information for the Invigilators: none

Information for Candidates

The trigonometric Fourier series of a periodic signal $x(t)$ of period $T_0 = 2\pi/\omega_0$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

with

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt, \quad a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t dt, \quad b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t dt.$$

The compact Fourier series is given by

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \quad \text{with} \quad C_0 = a_0, \quad C_n = \sqrt{a_n^2 + b_n^2}, \quad \theta_n = \tan^{-1}\frac{-b_n}{a_n}.$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{with} \quad D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

Some Fourier Trasforms

$$\cos\omega_0 t \iff \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\text{rect}(\tfrac{t}{\tau}) \iff \tau\text{sinc}(\tfrac{\omega\tau}{2})$$

$$\tfrac{W}{\pi}\text{sinc}(Wt) \iff \text{rect}(\tfrac{\omega}{2W})$$

Some useful trigonometric identities

$$\cos(x+y) = \cos x\cos y - \sin x \sin y$$

$$\sin x \cos y = \tfrac{1}{2}\sin(x-y) + \tfrac{1}{2}\sin(x+y)$$

$$\cos x \cos y = \tfrac{1}{2}\cos(x-y) + \tfrac{1}{2}\cos(x+y).$$

Euler's formula

$$e^{jx} = \cos x + j\sin x.$$

1. Consider the periodic signal $x(t)$ shown in Figure 1.

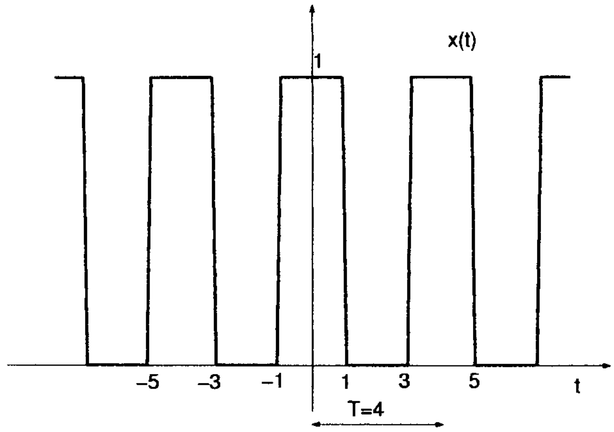

Figure 1: The periodic signal $x(t)$.

(a) Find the power of $x(t)$.

[4]

(b) Compute the trigonometric Fourier series of $x(t)$. That is, compute the coefficients a_0, a_n and b_n.

[4]

(c) Find the coefficients C_n and the phases θ_n of the compact Fourier series.

[4]

(d) Compute the coefficients D_n of the exponential Fourier series.

[4]

(e) The signal $x(t)$ is fed to a filter $h(t)$ giving output $y(t)$. The frequency response of the filter is

$$H(\omega) = \begin{cases} 1 & for \ \ |\omega| \leq 3 \text{ rad/s} \\ 0 & otherwise \end{cases}$$

Write the exact expression of the output $y(t)$.

[4]

2. Consider the energy signal $x(t) = \frac{10}{\pi}\text{sinc}(10t)$.

   (a) Sketch and dimension the Fourier transform of $x(t)$.

   [4]

   (b) Using Parseval's theorem compute the energy of $x(t)$.

   [4]

   (c) Sketch and dimension the spectrum of the DSB-SC modulated signal $s(t) = 2x(t)\cos 100t$.

   [4]

   (d) From the spectrum of $s(t)$, identify the upper sideband (USB) and the lower sideband (LSB) spectra.

   [4]

   (e) From the USB spectrum, write the exact expression of $\varphi_{USB}(t)$.

   [4]

3. Consider the power signal $x(t) = \cos 10t$.

(a) Find the power of $x(t)$.

[4]

(b) Compute the autocorrelation function of $x(t)$ defined as

$$\mathcal{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

[6]

(c) Determine the Power Spectral Density $(S_x(\omega))$ of $x(t)$,

[4]

(d) The signal $x(t)$ is fed to a filter with frequency response

$$H(\omega) = \frac{1}{1+j\omega}.$$

Compute the power P_y of the output signal $y(t)$.

[6]

4. Consider the frequency modulated signal

$$\varphi_{FM}(t) = A\cos\left[2\pi f_c t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha\right],$$

where the message signal is $m(t) = 100\ \mathrm{sinc}(100\pi t)$. The carrier is given by $c(t) = 4\cos(2\pi f_c t)$ with $f_c = 100$ MHz and $k_f = 20\pi$.

(a) Sketch and dimension the Fourier transform of $m(t)$.

[2]

(b) Determine the bandwidth of the baseband signal $m(t)$.

[2]

(c) Find the peak value m_p of the baseband signal.

[4]

(d) Using Carson's rule, determine the bandwidth of $\varphi_{FM}(t)$.

[4]

(e) Compute the average transmitted power.

[4]

(f) Using Carson's rule, compute the bandwidth of $\varphi_{FM}(t)$ if $m(t) = 100\mathrm{sinc}(200\pi t)$.

[4]

5. A sinusoidal source $v(t) = 10\sin(2\pi f_0 t)$ Volts with internal resistance $R = 50\ \Omega$ is connected to a transmission line having $L_0 = 0.25\ \mu\text{H/m}$ and $C_0 = 100\ \text{pF/m}$. The transmission line has length $L = 100$ m and is connected to a load Z_L (see Figure 2).

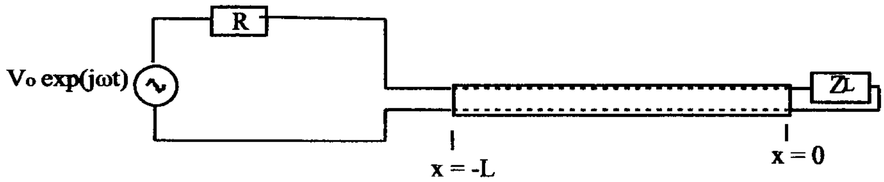

Figure 2: A transmission line connected to a sinusoidal source.

(a) Determine the phase velocity of the wave.

[4]

(b) Choose Z_L so that there is no reflection in the line.

[4]

(c) Assume $Z_L = 150\ \Omega$, compute the fraction of the incident power that is reflected at the load.

[4]

(d) Assume $Z_L = 0$ (short circuit termination),

i. find the lowest non-zero frequency at which $Z_{in} = 0$. (Recall that $Z_{in} = V(-L)/I(-L)$).

[4]

ii. find the lowest non-zero frequency at which the current flowing in the circuit is $i(t) = 0.2\sin(2\pi f_0 t)$ A.

[4]

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Exam Solutions

1. (a) Period of $x(t)$ is $T_0 = 4$.

$$P_x = \frac{1}{T_0}\int_{T_0} x^2(t)dt = \frac{1}{2}.$$

(b) $x(t)$ is an even function, therefore, $b_n = 0$.

$$a_0 = \frac{1}{4}\int_{-2}^{2} x(t)dt = \frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x(t)\cos n\omega_0 t dt = \int_{0}^{2} x(t)\cos n\omega_0 t dt = \int_{0}^{1} \cos n\omega_0 t dt = \frac{2}{n\pi}\sin(n\pi/2).$$

(c) $C_0 = a_0$, $C_n = \sqrt{a_n^2 + b_n^2} = |a_n|$, $\theta_n = 0$ if $a_n \geq 0$ $\theta_n = \pi$ otherwise .

(d)

$$D_n = \frac{C_n}{2}e^{j\theta_n},$$

$$D_{-n} = \frac{C_n}{2}e^{-j\theta_n},$$

$$D_0 = a_0.$$

(e) The filter cuts all the harmonics except for the fundamental one. Thus

$$y(t) = a_0 + a_1\cos(\pi t/2) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t/2).$$

2. (a)

$$\frac{10}{\pi}\text{sinc}(10t) \Longleftrightarrow \text{rect}(\omega/20),$$

(b) The energy of $x(t)$ is given by

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-10}^{10} dt = \frac{10}{\pi}$$

(c)

$$S(\omega) = \text{rect}(\frac{\omega - 100}{20}) + \text{rect}(\frac{\omega + 100}{20})$$

(d)

$$Y_{LSB}(\omega) = \text{rect}(\frac{\omega - 95}{10}) + \text{rect}(\frac{\omega + 95}{10}).$$

$$Y_{USB}(\omega) = \text{rect}(\frac{\omega - 105}{10}) + \text{rect}(\frac{\omega + 105}{10}).$$

(e)

$$\varphi_{USB}(t) = \frac{10}{\pi} sinc(5t) \cos(105t)$$

3. (a) $P_x = 1/2$

   (b)

$$\mathcal{R}_x(\tau) = \frac{1}{2} \cos 10\tau$$

   (c)

$$S_x(\omega) = \frac{\pi}{2}[\delta(\omega - 10) + \delta(\omega + 10)].$$

   (d)

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

   and

$$P_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_y(\omega) d\omega$$

   Thus,

$$P_y = \frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} [\delta(\omega - 10) + \delta(\omega + 10)] d\omega = 1/202$$

4. (a)

$$M(\omega) = \text{rect}\left(\frac{\omega}{200\pi}\right).$$

   (b) The bandwidth of the baseband signal is $B = 50$Hz.

   (c) The peak value of $m(t)$ is $m_p = m(0) = 100$.

   (d) Using Carson's rule the effective bandwidth is given by

$$B_{FM} = 2(\Delta f + B) = 2(\frac{k_f m_p}{2\pi} + B) = 2(1000 + 50) = 2100Hz.$$

   (e) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P_{FM} = A^2/2 = 16/2 = 8.$$

   (f) $B = 100$Hz but m_p is the same. Therefore

$$B_{FM} = 2(1000 + 100) = 2200Hz$$

5. (a) Phase velocity $u = 1/\sqrt{C_0 L_0} = 2 \times 10^8$m/s

   (b) The characteristic impedance of the line is

$$Z_0 = \sqrt{L_0/C_0} = \sqrt{0.25 \cdot 10^{-6}/100 \cdot 10^{-12}} = 50\Omega.$$

   Thus, there is no reflection if $Z_L = Z_0 = 50\Omega$

(c) The voltage reflection coefficient is $K_v = (Z_L - Z_0)/(Z_L + Z_0) = 1/2$. Thus, $K_p = 1/4$.

(d) i. If $Z_L = 0$ then $Z_{in} = v(-L)/I(-L) = jZ_0 \tan(kL)$; $Z_{in} = 0$ when $kL = \pi$. Thus $f_0 = u/2L = 10^6 = 1\text{MHz}$.

ii. $i(t) = v(t)/(Z_{in} + R)$. We want $i(t) = 0.2 \sin(2\pi f_0)$ A. Thus $Z_{in} + R = 50\Omega$, which implies $Z_{in} = 0$ and $f_0 = 1$ MHz.