MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

E1

PAPER 2

QUESTION

SOLUTION 1/1

Please write on this side only, legibly and neatly, between the margins

$$f(x,y) = y^4 + 4x^2y^2 - 2y^2 + 2x^2 - 1$$

$$\partial f/\partial x = 0, \quad \partial f/\partial y = 0 \Rightarrow$$

$$4x(2y^2+1) = 0$$

$$4y(y^2+2x^2-1) = 0$$ 2

From 1st equation $x = 0$. Sub into 2nd eqn $\Rightarrow$ $y = 0$, $y = \pm 1$. Hence stationary pts are $(0,0)$, $(0,1)$, $(0,-1)$ 2

To determine nature, require

$$f_{xx} = 4(2y^2+1)$$

$$f_{xy} = 16xy \qquad f_{xx} \equiv \partial^2 f/\partial x^2 \text{ etc}$$ 2

$$f_{yy} = 12y^2 + 8x^2 - 4$$

$(0,0)$: $f_{xx}(0,0) = 4$, $f_{xy}(0,0) = 0$, $f_{yy}(0,0) = -4$

$$\Delta = f_{xy}^2(0,0) - f_{xx}(0,0)f_{yy}(0,0) = 16 > 0$$ 2

$\therefore$ saddle.

$(0,\pm1)$: $f_{xx}(0,\pm1) = 12$, $f_{xy}(0,\pm1) = 0$

$f_{yy}(0,\pm1) = 8$

For either pt

$$\Delta = f_{xy}^2(0,\pm1) - f_{xx}(0,\pm1)f_{yy}(0,\pm1) = -96 < 0$$

Since f_{xx} and $f_{yy} > 0$ for both pts, each is a minimum 2,2

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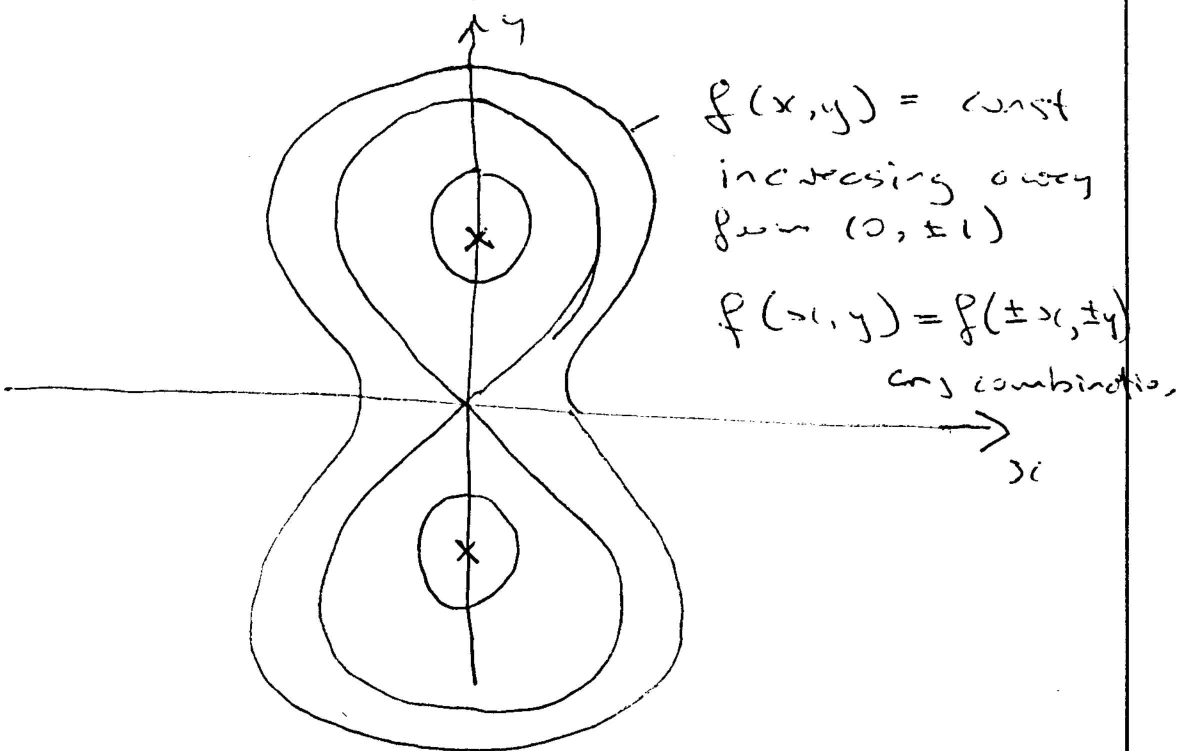
$f(0,0) = -1$, $f(0,\pm 1) = -2$

On $y = 0$, $f(x,0) = 2x^2 - 1$

On $x = 0$, $f(0,y) = y^4 - 2y^2 - 1$

$= (y^2-1)^2 - 2$

Hence, contour plot looks like



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Setter : J ELGIN
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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

E 2

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PAPER 2

QUESTION

SOLUTION 2/1

$$x = r\cosh\theta \quad , \quad y = r\sinh\theta$$

$$\Rightarrow \quad \sqrt{x^2-y^2} = r \quad , \quad \tanh^{-1}(y/x) = \theta$$

Then, $\dfrac{\partial r}{\partial x} = x/r \;,\; \dfrac{\partial r}{\partial y} = -y/r$

$$\partial\theta/\partial x = -y/r^2 \;,\; \partial\theta/\partial y = x/r^2$$ 4

$$\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial f}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial f}{\partial r} - y/r^2\,\frac{\partial f}{\partial \theta}$$

$$= \cosh\theta\,\frac{\partial f}{\partial r} - \frac{\sinh\theta}{r}\frac{\partial f}{\partial \theta}.$$ 2

Similarly,

$$\frac{\partial f}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial f}{\partial \theta}$$

$$= \frac{-y}{r}\frac{\partial f}{\partial r} + \frac{x}{r^2}\frac{\partial f}{\partial \theta}$$ 3

$$= -\sinh\theta\,\frac{\partial f}{\partial r} + \frac{\cosh\theta}{r}\frac{\partial f}{\partial \theta}.$$

ii) With $f = \sqrt{x^2-y^2}\,\tanh^{-1}(y/x) = r\theta$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2-y^2}}\tanh^{-1} y/x + \sqrt{x^2-y^2}\left(-\frac{y}{x^2-y^2}\right)$$

$$= \frac{x}{\sqrt{x^2-y^2}}\tanh^{-1} y/x - \frac{y}{\sqrt{x^2-y^2}}$$

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Checker : HERBERT

Checker's signature : [signature]

$= \theta \cosh\theta - \sinh\theta$

Also, $\frac{\partial f}{\partial \theta} = r, \quad \frac{\partial f}{\partial r} = \theta$

$\therefore \cosh\theta \frac{\partial f}{\partial r} - \frac{\sinh\theta}{r}\frac{\partial f}{\partial \theta} = \theta\cosh\theta - \sinh\theta$

$\therefore \quad \frac{\partial f}{\partial x} = \cosh\theta \frac{\partial f}{\partial r} - \frac{\sinh\theta}{r}\frac{\partial f}{\partial \theta}.$ 3

Similarly, $\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{x^2-y^2}}\tanh^{-1}y/x + \sqrt{x^2-y^2}\,\frac{x}{x^2-y^2}$

$= \frac{-y}{\sqrt{x^2-y^2}}\tanh^{-1}\left(\frac{y}{x}\right) + \frac{x}{\sqrt{x^2-y^2}}$

$= -\theta\sinh\theta + \cosh\theta.$

Also $-\sinh\theta\frac{\partial f}{\partial r} + \frac{\cosh\theta}{r}\frac{\partial f}{\partial \theta}$

$= -\theta\sinh\theta + \frac{\cosh\theta}{r}.r$

$= \cosh\theta - \theta\sinh\theta.$

ie, $\frac{\partial f}{\partial y} = -\sinh\theta\frac{\partial f}{\partial r} + \frac{\cosh\theta}{r}\frac{\partial f}{\partial \theta}$ 3

15

E 3

PAPER 2

QUESTION

SOLUTION 3

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$$I_2 = \frac{\pi/4 - 0}{2}\left[0 + \pi/4 \sin \pi/4\right]$$

$$= \pi/8 \left[0 + \pi/4 \sin \pi/4\right] = 0.2180$$ (3)

$$I_3 = \frac{\pi/8 - 0}{2}\left[0 + \pi/8 \sin \pi/8\right] + \frac{\pi/8}{2}\left[\frac{\pi}{8}\sin \pi/8 + \frac{\pi}{4}\sin\frac{\pi}{4}\right]$$

$$= \frac{\pi}{16}\left[2 \times 0.1503 + 0.554\right] = 0.1680$$ (3)

If we were to use Simpson's Rule immediately

$$I = \frac{\pi}{24}\left[f(0) + 4f(\pi/8) + f(\pi/4)\right]$$

$$= \frac{\pi}{24}\left[0 + 4 \times (0.1503) + 0.5554\right] = 0.1513$$ (3)

If we extrapolate the Trapezium Rule we have

$$I_R = (4I_3 - I_2)/3 = 0.1513$$ (3)

This agrees with the Simpson Rule approximation

To find the true solution we integrate by parts

$$I_{True} = -\left[\theta\cos\theta\right]_0^{\pi/4} + \int_0^{\pi/4} \cos\theta \, d\theta$$

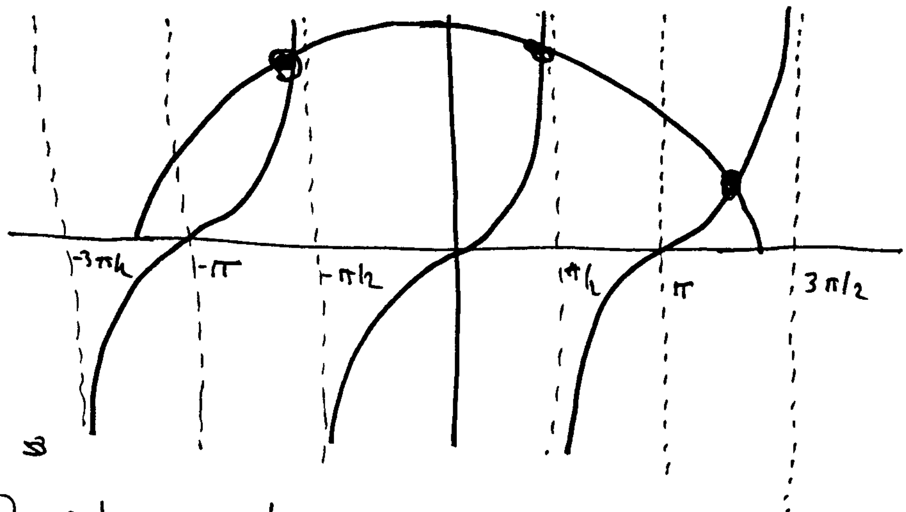
$$= -\left[\theta\cos\theta\right]_0^{\pi/4} + \left[\sin\theta\right]_0^{\pi/4}.$$

$$= (1 - \pi/4)\cdot 1/\sqrt{2} = 0.1517$$ (3)

Setter : CASH Setter's signature : JL Cash

Checker : C STRIDLER-ROWE Checker's signature : C Stridler Rowe

EXAMINATION QUESTIONS/SOLUTIONS	SESSION 2002-2003	COURSE
Setters are advised that Checkers, Editors, Typists and External Examiners greatly appreciate the merits of accuracy, legibility and neatness.		SETTER CASH
Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please		QUESTION NO.
		SOLUTION NO. E4
		MARKSCHEME



(4)

Rewrite equation as

$$0 = f(x) = \tan x - \sqrt{a^2 - x^2}$$

Then $f'(x) = \sec^2 x + \dfrac{x}{\sqrt{a^2-x^2}}$.

So Newton-Raphson is

$$x_{n+1} = x_n - \frac{\left(\tan x_n - \sqrt{16 - x_n^2}\right)}{\sec^2 x_n + \dfrac{x_n}{\sqrt{16-x_n^2}}}$$

(3)

Iterates are $x_0 = 1.3$, $x_1 = 1.31262$, $x_2 = 1.312086$, $x_3 = 1.31208 = x_4$ (4)

$x_0 = 3.8$, $x_1 = 3.90245$, $x_2 = 3.89079$, $x_3 = 3.89049 = x_4$ (4)

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

E 5

PAPER 2

QUESTION

SOLUTION 6/1

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Std equation of plane is

$$\underset{\sim}{n} \cdot \underset{\sim}{r} = n_1 x + n_2 y + n_3 z = p$$

where $\underset{\sim}{n}$ is unit vector $\perp$r to plane, p is perpendicular distance from origin.

i) Plane $x + y - 2z = 3$ can be written

$$\frac{x}{(1^2+1^2+(-2)^2)} + \frac{y}{1^2+1^2+(-2)^2} + \frac{-2z}{1^2+1^2+(-2)^2} = \frac{3}{1^2+1^2+(-2)^2)}$$

or $\underset{\sim}{n}_1 \cdot \underset{\sim}{r} = \frac{3}{\sqrt{6}} = \sqrt{3/2} = p_1$ 2

Plane $2x - 2y + z = 1$ can be written

$$\underset{\sim}{n}_2 \cdot \underset{\sim}{r} = \frac{1}{\sqrt{9}} = \frac{1}{3} = p_2$$ 2

Hence $p_1 = \sqrt{3/2}$, $p_2 = 1/3$.

ii) Let direction cosine of req'd st. line be (ℓ, m, n). Since line lies in both planes, it must be $\perp$r to $\underset{\sim}{n}_1$ & $\underset{\sim}{n}_2$ $\Rightarrow$

$$\ell + m - 2n = 0$$
$$2\ell - 2m + n = 0$$

leading to $\frac{\ell}{3} = \frac{m}{5} = \frac{n}{4}$ 6

The direction ratios are therefore $(3, 5, 4)$

Setter : J ELGIN Setter's signature : J Elgin

Checker : F. BERKSHIRE Checker's signature : [signature] and hence

$$\underset{\sim}{t} = \frac{1}{\sqrt{50}}[3, 5, 4]$$

is a unit vector along req'd line.

Pt. on line: This will also lie on both planes. Put $x = 0$ in ~~eq~~ original equs for planes $\Rightarrow$

$$\begin{aligned} y - 2z &= 3 \\ -2y + z &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y &= -5/3 \\ z &= -7/3 \end{aligned}$$ 3

Req'd vector to this pt on line is

$$\underset{\sim}{a} = [0, -5/3, -7/3]$$

Hence, req'd vector equation for line is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{t}, \text{ parameter } \lambda$$ 2

15

(i) $A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$. 1

So $A^2 = I$ is equivalent to

$$\left.\begin{aligned} a^2+bc &= d^2+bc = 1 \\ b(a+d) &= c(a+d) = 0 \end{aligned}\right\} (*)$$ 1

(a) If $b = c = 0$, $(*)$ reduces to $a^2 = d^2 = 1$, giving the four solutions

$A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. 2

(b) If neither b nor c is 0, $(*)$ becomes $d = -a$, $1 - a^2 = bc \neq 0$, giving the solutions $A = \begin{pmatrix} a & b \\ (1-a^2)/b & -a \end{pmatrix}$, 3

$a \neq \pm 1$, b arbitrary $\neq 0$.

(c) If $b \neq 0$, $c = 0$, $(*)$ becomes

PAPER 2
QUESTION
SOLUTION 7 (cont).

$d = -a$, $a^2 = 1$, giving solutions

$A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$, b arbitrary.

Similarly if $b = 0$, $c \neq 0$ we get

$A = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}$, c arbitrary — 3

(ii) (a) Not valid, because $(A+B)(A-B)$
$= A^2 + BA - AB - B^2$, and in general
$BA \neq AB$. — 2

(b) Valid, because $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$
$= AIA^{-1} = AA^{-1} = I$, and similarly
$(B^{-1}A^{-1})(AB) = I$. — 2

(c) Not valid even for scalars: $A = B = I$
is a counterexample. — 1

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Setter: WILSON — Setter's signature: J. Wilson
Checker: — Checker's signature:

PAPER 2

QUESTION

SOLUTION 8

Denoting the missing elements as l_{ij} and U_{ij} we have.

$U_{11} = 1,\ U_{12} = 1,\ U_{22} = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ \cdot & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & \cdot & \cdot \\ 0 & 0 & \cdot \end{pmatrix}.$$

So $l_{21} = 1,\quad 1 + U_{22} = a \Rightarrow U_{22} = a-1,\quad 1 + U_{23} = 2 \quad U_{23} = 1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & \cdot \end{pmatrix}$$

Finally $l_{31} = 1,\quad l_{31} + (a-1)l_{32} = a \Rightarrow l_{32} = 1 \quad 2 + U_{33} = 2a \quad U_{33} = 2a-2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & 2a-2 \end{pmatrix}$$ (5)

So we need to solve $LU\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$

Put $U\underline{x} = y$ so $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix};\ y = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$ (2)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 1 \\ 0 & 0 & 2a-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$x_3 = b/(2a-2)$

$(a-1)x_2 + b/2(a-1) = 0$

$\therefore x_2 = -b/2(a-1)^2$

$x_1 - b/2(a-1)^2 + b/2(a-1) = 0$

$x_1 = \dfrac{b}{2(a-1)}\left[\dfrac{1}{a-1} - 1\right] = \dfrac{b(2-a)}{2(a-1)^2}$ (4)

If $a=1,\ b \neq 0$ No solution (1)

If $a=1,\ b=0$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore$ $x_3 = 0$, $x_1 = -x_2$; infinitely many solutions. (3)

Setter : CASM

Setter's signature :

Checker : RIDLER-ROWE

Checker's signature : C Ridler Rowe

(i) $(y^2 - x^2)\dfrac{dy}{dx} = x^2 + 2xy \rightarrow \dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{(y^2 - x^2)} = \dfrac{1 + 2\frac{y}{x}}{\left(\frac{y}{x}\right)^2 - 1}$ 1

$\therefore$ Put $y(x) = xv(x) \rightarrow x\dfrac{dv}{dx} + v = \dfrac{1 + 2v}{v^2 - 1}$ 1, 1

$\therefore \quad x\dfrac{dv}{dx} = \dfrac{1 + 2v}{v^2 - 1} - v = \dfrac{1 + 2v - v^3 + v}{v^2 - 1} = \dfrac{1 + 3v - v^3}{v^2 - 1}$ 2

$\therefore \quad \displaystyle\int \frac{dx}{x} = \int \frac{v^2 - 1}{1 + 3v - v^3}\, dv \quad \therefore \quad \ln x = -\frac{1}{3}\ln(1 + 3v - v^3) + c_1$ 2

$\rightarrow \quad \ln(1 + 3v - v^3)x^3 = c_2. \quad \therefore \quad x^3 + 3x^2y - y^3 = c_3.$

Given that $y(1) = 2$, $c_3 = 1 + 6 - 8 = -1$. 1

$\therefore$ Required solution is : $y^3 - x^3 - 3x^2y = 1$ 1

(ii) $(x+1)\dfrac{dy}{dx} - 3y = (x+1)^5 \rightarrow \dfrac{dy}{dx} - \dfrac{3}{(x+1)}y = (x+1)^4.$ 1

Integrating factor : $\exp\left(-\displaystyle\int \frac{3\,dx}{(x+1)}\right) = \exp(-3\ln(x+1)) = (x+1)^{-3}.$ 1, 1

$\therefore$ ODE $\rightarrow \dfrac{d((x+1)^{-3}y)}{dx} = x + 1. \quad \therefore (x+1)^{-3}y = \dfrac{1}{2}(x+1)^2 + c$ 1, 1

$\therefore$ The general solution is $y(x) = \dfrac{1}{2}(x+1)^5 + c(x+1)^3$

$= (x+1)^3\left(\dfrac{1}{2}x^2 + x + c_1\right)$ 1

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Setter : HERBERT Setter's signature : [signature]

Checker : REICH Checker's signature : [signature]

(i) The auxiliary equation is $m^2+m-2=0$. Hence $m_1=1$, $m_2=-2$. The general solution of the homogeneous equation is ae^x+be^{-2x}. 3 mark

To find a solution of the inhomogeneous equation try $y=\alpha x+\beta$. Then we get

$\alpha - 2\alpha x - 2\beta = x$. Whence $\alpha=-\frac{1}{2}, \beta=-\frac{1}{4}$. 3 marks

The answer is $ae^x+be^{-2x}-\frac{1}{2}x-\frac{1}{4}$.

(ii) The auxiliary equation is $m^2+2m=0$. The general solution of the homogeneous equation is thus $a+be^{-2x}$. Now try 3 marks

$y(x)=\alpha\cos x+\beta\sin x$. We obtain

$-\alpha\cos x-\beta\sin x+2\alpha\sin x+2\beta\cos x=\cos x$.

Hence $-\alpha+2\beta=1$, $\beta+2\alpha=0$. We get

$\alpha=-\frac{1}{5}$, $\beta=\frac{2}{5}$. The general solution is 3 marks

$$a+be^{-2x} - \frac{1}{5}\cos x+\frac{2}{5}\sin x.$$

The initial condition says that

$a+b-\frac{1}{5}=1$, $-2b+\frac{2}{5}=0$, whence $b=\frac{1}{5}$,

$a=1$. The answer is 3 marks

$$1+\frac{1}{5}e^{-2x}-\frac{1}{5}\cos x+\frac{2}{5}\sin x.$$

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Checker: SL Checker's signature: [signature]

a) Identify as Fourier Cosine series $\therefore b_n = 0$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left(\frac{x\sin nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right)$$

$$= \frac{2}{\pi}\left[\frac{-1+(-1)^n}{n^2}\right] = \begin{cases} \frac{-4}{\pi n^2}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

$$a_0 = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$\therefore x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n,\text{odd}} \frac{\cos nx}{n^2}, \quad 0 \leq x < \pi$$ 6

b) Fourier Sine series $\therefore a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx = \frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$= -\frac{2}{\pi}\frac{\pi(-1)^n}{n}$$

$$\therefore x = -2\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n}, \quad 0 \leq x < \pi$$ 5

In a), even fn is continuous at $x=0$.
Put $x = 0$

c)
$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n,\text{odd}}\frac{1}{n^2}$$

$$\therefore \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$ 4

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Checker : F. BERKSHIRE Checker's signature : [signature]

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \mathbf{b}\,.\,\mathbf{c} \times \mathbf{a} = \mathbf{c}\,.\,\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b\,;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i \sinh z\,; \quad \sinh iz = i \sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \dots + \binom{n}{r} D^r f\,D^{n-r} g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1} f^{(n+1)}(a + \theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.