Master

UNIVERSITY OF LONDON

[E1.14 (Maths 2) 2007]

B.ENG. AND M.ENG. EXAMINATIONS 2007

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 31st May 2007 10.00 am - 1.00 pm

Answer EIGHT questions.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Let H denote the Heaviside function

$$H(x) = \begin{cases} 1 & : \quad x \geq 0, \\ 0 & : \quad x < 0. \end{cases}$$

Sketch a graph of the following functions, stating whether each is even or odd:

(i) $H(x) - H(x-1)$,

(ii) $H(x-1/2) - H(x+1/2)$,

(iii) $x^2\,[\,H(x-1/2) - H(x+1/2)\,]$.

Sketch a graph of the following functions, giving a precise domain and range in each case:

(iv) $x/(x+1)$,

(v) $x^2 \sin(1/x)$.

Finally, evaluate the limit

$$\lim_{x \to 0} \frac{\tan(px)}{\tan(qx)}\,.$$

2. Consider the function

$$f(x) = (x^2 - 1)^2.$$

Find the stationary points of f and provide the details of the calculation that determines their nature. Hence draw a sketch of f on the interval $[-2, 2]$, noting all local and global extrema and any other important features.

Use the information contained in your first graph to sketch a graph of the function $e^{-f(x)}$ on $[-2, 2]$.

PLEASE TURN OVER

3. A curve Γ is defined parametrically by the two functions

$$x(t) \ = \ 1 \ + \ \cos t, \quad y(t) \ = \ t \ + \ \sin t \ .$$

Find dx/dt and dy/dt and hence give the length of Γ between the points $t = 0$ and $t = 1/10$. Show that $1/5$ units is an approximation of this length, but that $2399/12000$ units is a better approximation.

4. (i) Integrate $e^x \cos x$, omitting the arbitrary constant of integration.

(ii) You are given that $\dfrac{d}{dx} \sin x = \cos x$ and $\sin^{-1} x$ denotes the inverse function to $\sin x$ on a certain domain. Use these two pieces of information to deduce that

$$\frac{d}{dx} \sin^{-1} x \ = \ \frac{1}{\sqrt{1 - x^2}}$$

for all x in that domain. You must clearly state any properties of trigonometric functions that you use.

(iii) Use a power series representation of $\dfrac{1}{1 - x}$ that is convergent for $-1 < x < 1$ to deduce a power series representation of $\ln \mid 1 - x \mid$ that converges for $-1 \leq x < 1$.

Hence deduce that

$$\ln(1/2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

stating clearly any convergence tests that you use.

5. Using the ratio test for convergence, or otherwise, deduce for what values of x each of the following series converges :

(i) $\sum_{n=1}^{\infty} a_n$ with $a_n = \dfrac{(2x)^n}{n(n+1)}$,

(ii) $\sum_{n=1}^{\infty} b_n$ with $b_n = \dfrac{x^{2n}}{(3n)\,!}$,

(iii) $\sum_{n=1}^{\infty} c_n$ with $c_n = (x(x-1))^n/e^n$.

Use the integral test to find numbers m and M such that

$$m \le \sum_{n=1}^{\infty} n^{-3} \le M .$$

6. You are given the integral

$$\int e^{kx} dx = \frac{e^{kx}}{k} + \text{Const}$$

which holds for all complex k and real x. Use this to deduce the values of

$$a_n = \int_{-\pi}^{\pi} e^x \cos nx \, dx \quad \text{and} \quad b_n = \int_{-\pi}^{\pi} e^x \sin nx \, dx$$

in terms of $\sinh \pi \equiv \frac{1}{2}(e^{\pi} - e^{-\pi})$ where n is a fixed *integer*.

The complex Fourier series representation of the real 2π-periodic function $f(x)$ on $(-\pi, \pi)$ is given by

$$f(x) = c_0 + 2\sum_{n=1}^{\infty} Re(c_n e^{inx}) \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Evaluate the coefficients c_n when $f(x)$ coincides with e^x on $(-\pi, \pi)$ and hence evaluate

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

Hint: Use the following version of Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 \, dx = c_0^2 + 2\sum_{n=1}^{\infty} |c_n|^2 .$$

to first evaluate

$$\sum_{n=1}^{\infty} \frac{1+n^2}{(1+n^2)^2}.$$

PLEASE TURN OVER

7. (i) If $z = f(s)$ where $s = \dfrac{x+y}{xy}$ show that

$$x^2 \frac{\partial z}{\partial x} - y^2 \frac{\partial z}{\partial y} = 0 .$$

(ii) The height of a cylindrical tube with radius r and volume V is found by the formula $V = \pi r^2 h$. Given that the percentage errors in the measurements of r and V are at most 0.4% and 0.6%, respectively, give an upper bound on the percentage error committed in the calculation of the height.

8. (i) Consider the equation

$$x^2 - x - 6 = 0.$$

Show that following are iteration schemes for computing the two solutions of this equation:

(a) $x_{n+1} = f(x_n), \quad f(x) = \sqrt{x+6},$

(b) $x_{n+1} = h(x_n), \quad h(x) = \dfrac{6}{x-1},$

(c) $x_{n+1} = g(x_n), \quad g(x) = x^2 - 6,$

Show that iteration scheme (a) converges to the positive solution of the equation, iteration scheme (b) converges to the negative solution and that iteration scheme (c) does not converge to any of the solutions.

Do not attempt to carry out the iteration in any of these cases.

(ii) Show that the equation

$$x^2 + x - \cos x = 0$$

has a unique solution for $x \in [0, \pi/2]$. Write down the Newton-Raphson scheme for computing this solution. Starting at $x_0 = 0.5$, compute the first two Newton-Raphson iterates.

9. (i) Find the general solution to the differential equation

$$\frac{dy}{dx} + \frac{4}{x}y = x^4$$

using the integrating factor method, or otherwise.

(ii) Show that the differential equation

$$(x\cos y - 2y)\frac{dy}{dx} + (x + \sin y) = 0$$

is exact and obtain its general solution in implicit form.

10. Consider the function f defined on $[-\pi, \pi)$

$$f(x) = \left\{ x \;:\; -\pi \le x < \pi, \right.$$

that is periodically extended to all real x. Find a real Fourier series for f and use Parseval's theorem to evaluate

$$\sum_{n=1}^{\infty} n^{-2}.$$

To what values does the Fourier series converge at $x = -\pi$ and $x = +\pi$?

Use either the real form of Parseval's theorem given in the formulae sheet or the complex form given in Q6.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\,\mathbf{b},\,\mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \dots + \tbinom{n}{r} D^r f\,D^{n-r}g + \dots + D^n f\, g\,.$$

(b) Taylor's expansion of $f(x)$ about $x=a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0<\theta<1\,.$

(c) Taylor's expansion of $f(x,\,y)$ about $(a,\,b)$:

$$f(a+h,\,b+k) = f(a,\,b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a.b} + \dots$$

(d) Partial differentiation of $f(x,\,y)$:

i. If $y=y(x)$, then $f=F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x=x(t)$, $y=y(t)$, then $f=F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x=x(u,\,v)$, $y=y(u,\,v)$, then $f=F(u,\,v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,,\qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\,y)$ occur where $f_x=0,\ f_y=0$ simultaneously.
Let $(a,\,b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D>0$ and $f_{xx}(a,\,b)<0$, then $(a,\,b)$ is a maximum;
If $D>0$ and $f_{xx}(a,\,b)>0$, then $(a,\,b)$ is a minimum;
If $D<0$ then $(a,\,b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\,y)dx + Q(x,\,y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

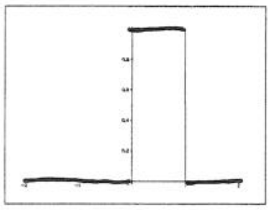
EE I (2) (1)

SOLUTIONS

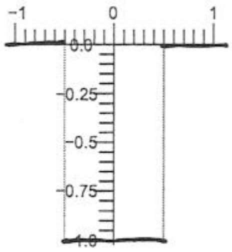
1. (a) neither even nor odd (b) even (c) even (d) domain is $\{x \in \mathbb{R} : x \neq -1\}$ and range $\{x \in \mathbb{R} : x \neq 1\}$ (e) domain and range is $\mathbb{R}$, although $\{x \in \mathbb{R} : x \neq 0\}$ is a reasonable domain with the same range.

5

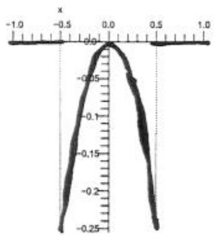
(i)



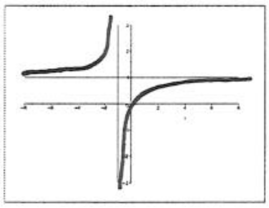
(ii)

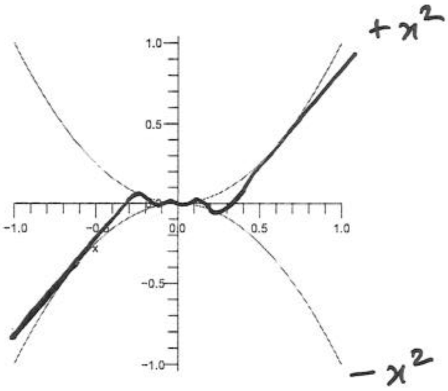


(iii)



(iv)





(v)

$10 = 5 \times 2$

In order to evaluate the limit [noting L'hôpital's rule would also receive fulls marks.]

$$\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)},$$

Setter

R. Beardmore

Checker

G. Pavliotis

write this as

$$\lim_{x\to 0}\frac{\sin(px)}{\cos(px)}\frac{\cos(qx)}{\sin(qx)},$$

which equals

$$\frac{p}{q}\lim_{x\to 0}\frac{\sin(px)}{px\cos(px)}\frac{qx\cos(qx)}{\sin(qx)},$$

or

$$\frac{p}{q}\lim_{x\to 0}\frac{\sin(px)}{px}\lim_{x\to 0}\frac{\cos(px)}{\cos(qx)}\lim_{x\to 0}\frac{qx}{\sin(qx)}=\frac{p}{q}.$$

5

Q From tutorial sheet

Setter

R. Boardman

Checker

G. Pavlidis

IEE (2) 2

2. If $f(x) = (x^2-1)^2$ then f is an even function

$$\frac{df}{dx} = 2(x^2-1)\cdot 2x$$

and

$$\frac{d^2f}{dx^2} = 2(x^2-1)\cdot 2 + 4x\cdot 2x.$$

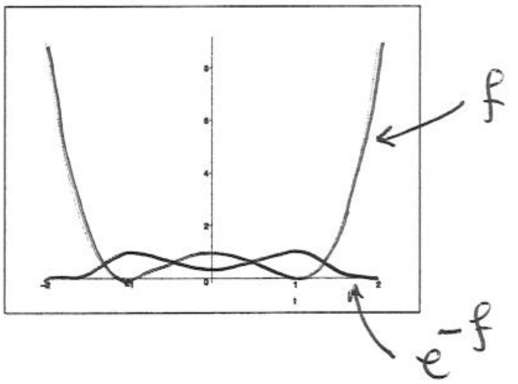
Hence $f'(x) = 0$ when $x = \pm 1, x = 0$ and

$$\left.\frac{d^2f}{dx^2}\right|_{x=\pm 1} = 8 > 0$$

so that $x = \pm 1$ are both local minima. At $x = 0$, $f''(0) = -4 < 0$ so that $x = 0$ is a local maximum. 10

At the boundaries $x = \pm 2$ we have $f(\pm 2) = (4-1)^2 = 9$ and $f(0) = 1$ so that global maxima are obtained at $x = \pm 2$, global minima are attained at $x = \pm 1$ as $f(x) \geq 0$ for all x. 2

Now $e^{-f(x)}$ is also even and has minima where f has maxima and vice versa as shown in the plot below. 4



4

standard plotting question.

R B...

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I EE (2) 3

3. Using the information given we have

$$\frac{dx}{dt} = -\sin(t),\ \frac{dy}{dt} = 1 + \cos(t),$$

and so the required length is

$$\ell := \int_0^{1/10} \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt = \int_0^{1/10} \left(\sin^2(t) + (1+\cos(t))^2\right)^{1/2} dt$$

$$= \int_0^{1/10} \left(\sin^2(t) + \cos^2(t) + 1 + 2\cos(t)\right)^{1/2} dt = \int_0^{1/10} (2 + 2\cos(t))^{1/2} dt$$

$$= \sqrt{2} \int_0^{1/10} (1+\cos(t))^{1/2} dt$$ 10

but $\cos(t) = \cos(t/2 + t/2) = \cos^2(t/2) - \sin^2(t/2) = 2\cos^2(t/2) - 1$ and so

$$\ell = \sqrt{2} \int_0^{1/10} \left(2\cos^2(t/2)\right)^{1/2} dt = 2\int_0^{1/10} |\cos(t/2)|\, dt = 2\int_0^{1/10} \cos(t/2) dt$$

on this region. Hence

$$\ell = 2\int_0^{1/10} \cos(t/2) dt = 4[\sin(t/2)]_0^{1/10} = 4\sin(1/20).$$ 5

The first term of the Maclaurin series expansion of the sine function gives $\sin(x) \simeq x$ and so $4/20 = 1/5$ is a first-order approximation. A better approximation comes about from using

$$\sin(x) \simeq x - \frac{x^3}{3!}$$

so that $4\sin(1/20) \simeq 4\left(1/20 - \frac{(1/20)^3}{3!}\right) = \frac{1}{5} - \frac{1}{12000} = 2399/12000.$ 5

Similar on Q sheet

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IEE (2) 4

4. (a) This answer uses $\int e^x \cos(x)dx = Re\left(\int e^x e^{ix} dx\right)$ (although integration by parts is also acceptable and will receive full marks). Now

Simulents Q sheet

$$Re\left(\int e^x e^{ix} dx\right)$$

$$= Re\left(\int e^{(1+i)x} dx\right) = Re\left(\frac{e^{(1+i)x}}{1+i}\right) + C = Re\left(\frac{e^{(1+i)x}}{1+i} \times \frac{1-i}{1-i}\right) + C$$

$$= Re\left(\frac{e^{(1+i)x}}{2}(1-i)\right) + C = \frac{e^x}{2} Re\left(e^{ix}(1-i)\right) + C$$

$$= \frac{e^x}{2}(\cos(x) + \sin(x)) + C.$$

6

Here C denotes an arbitrary real constant.

(b) The required derivative can be obtained from

$$\sin^{-1}(\sin(x)) = x$$

for all x in a suitable set. Differentiating this gives

$$\frac{d}{dx}(\sin^{-1})(\sin(x)) \frac{d}{dx}(\sin(x)) = 1$$

and so

$$\frac{d}{dx}(\sin^{-1})(\sin(x)) \cos(x) = 1$$

or

$$\frac{d}{dx}(\sin^{-1})(\sin(x)) = 1/\cos(x).$$

Using $\sin^2(x) + \cos^2(x) = 1$ for all x on setting $y = \sin(x)$ we obtain $\cos(x) = \pm\sqrt{1-y^2}$ and so the domain in the question can be chosen such that

$$\frac{d}{dx}(\sin^{-1})(y) = 1/\sqrt{1-y^2}.$$

6

(c) Using

$$1 + x + x^2 + ... = \frac{1}{1-x}$$

that converges for all $|x| < 1$ we can integrate this term-by-term to give

$$C + x + \frac{x^2}{2} + \frac{x^3}{3} + ... = C + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

R. B[illegible]

[illegible]

4. (Cont.) which equals

$$\int \frac{1}{1-x} dx = -\ln|1-x|,$$

for some constant of integration C. Setting $x = 0$ into this summation gives $C = 0$ as $\ln(1) = 0$ and so for $|x| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln|1-x|,$$ 5

Put $x = -1$ in above series expansion to get

$$-\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ 2

Thus $\ln \frac{1}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Note that the series converges when $x = -1$ by alternating series test. 1

R Bl[illegible] [illegible]

I (EE) (2) 5

5. (a) We compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(n+1)(n+2)} \times \frac{n(n+1)}{(2x)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{2x}{(n+2)} \times n\right| = |2x| \lim_{n\to\infty}\left|\frac{n}{n+2}\right| = |2x|.$$

The limit ratio test now tells us that the given series converges for $|x| < 1/2$ and diverges for $|x| > 1/2$. On setting $x = 1/2$ we see that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges by a comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which has been shown to converge in lectures using the integral test. Setting $x = -1/2$ gives the same conclusion (possibly using the Alternating Series Test) and so the region of convergence is $-1/2 \le x \le 1/2$. 5

(b) Computing $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{3(n+1)\,!} \times \frac{(3n)\,!}{x^{2n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x^2}{3(n+1)\,!} \times (3n)\,!\right| = |x|^2 \lim_{n\to\infty}\left|\frac{(3n)\,!}{3(n+1)\,!}\right|$$

which equals

$$|x|^2 \lim_{n\to\infty}\left|\frac{1}{(3n+3)(3n+2)(3n+1)}\right| = 0 < 1.$$

5

Hence the series converges for all real x.

(c) Computing $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x(x-1))^{n+1}}{e^{n+1}} \times \frac{e^n}{(x(x-1))^n}\right| = \left|\frac{x(x-1)}{e}\right|.$$

The ratio test gives convergence for $|x(x-1)| < e$ and divergence for $|x(x-1)| \ge e$, where the last non-strict inequality is obtained from inspection of convergence properties at the boundaries of the region specified by the ratio test. 5 seen

In order to find m and M such that $m \le \sum_{n=1}^{\infty} n^{-3} \le M$ clearly the first term $m = 1$ works, as does the sum of the first two terms $m = 1 + 1/8 = 9/8$. To find a suitable M note that

$$\sum_{n=2}^{\infty} n^{-3} \le \int_1^{\infty} x^{-3}dx = \left[-\frac{1}{2}x^{-2}\right]_1^{\infty} = 1/2$$

R Bl

[illegible]

IEE (2) 5

5. (Cont.) from the integral test and so $M = 1/2 + 1 = 3/2$ is an upper bound on the sum.

[An M that is proposed as an upper bound but with no rationale will receive zero marks, even if it *is* an upper bound. Careful use of the comparison test, comparing with a series like $\sum n^{-2}$ should receive full marks.] 5

Done in lectures.

R Bell

[signature]

IEE (2) 6

6. To compute the sequences (a_n) and (b_n) let us compute

$$c_{-n} := \int_{-\pi}^{\pi} e^x e^{inx} dx = \int_{-\pi}^{\pi} e^{(in+1)x} dx = \left[\frac{e^{(in+1)x}}{1+in}\right]_{-\pi}^{\pi}$$

2

$$= \left[\frac{e^{(in+1)x}(1-in)}{1+n^2}\right]_{-\pi}^{\pi} = \frac{1}{1+n^2}\left[e^{(in+1)x}(1-in)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{1+n^2}\left[\left(e^{(in+1)\pi} - e^{-(in+1)\pi}\right)(1-in)\right]$$

$$= \frac{1}{1+n^2}\left[(-1)^n\left(e^{\pi} - e^{-\pi}\right)(1-in)\right]$$

$$= 2\sinh(\pi)\frac{(-1)^n}{1+n^2} + 2i\sinh(\pi)\frac{(-1)^{n+1}n}{1+n^2} = a_n + ib_n.$$

From the above computation we have

$$c_{-n} = \frac{1}{2\pi}\frac{(-1)^n}{1+n^2}\left(e^{\pi} - e^{-\pi}\right)(1-in) = \frac{\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}(1-in)$$

and so

$$c_n = \frac{\sinh(\pi)}{\pi}\frac{(-1)^n}{1+n^2}(1+in).$$

10

Using the given version of Parseval's formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x} dx = c_0^2 + 2\sum_{n=1}^{\infty}|c_n|^2 = \left(\frac{\sinh(\pi)}{\pi}\right)^2\left(1 + 2\sum_{n=1}^{\infty}\frac{1+n^2}{(1+n^2)^2}\right)$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2x} dx = \frac{1}{2\pi}\left[\frac{1}{2}e^{2x}\right]_{-\pi}^{\pi} = \frac{\sinh(2\pi)}{2\pi}$$

we finally obtain

$$\frac{\sinh(2\pi)}{2\pi} = \left(\frac{\sinh(\pi)}{\pi}\right)^2\left(1 + 2\sum_{n=1}^{\infty}\frac{1+n^2}{(1+n^2)^2}\right)$$

and so

$$\sum_{n=1}^{\infty}\frac{1+n^2}{(1+n^2)^2} = \frac{1}{2}\left(\frac{\pi\sinh(2\pi)}{2(\sinh(\pi))^2} - 1\right).$$

8

Many egs done in tutorials of complex F-series

R B [signature]

[signature]

IEE (2) 7

7. (i) Let $u = \frac{x+y}{xy}$. We have that

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{df}{du}\frac{\partial u}{\partial x} \\ &= -\frac{1}{x^2}\frac{df}{du}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{df}{du}\frac{\partial u}{\partial y} \\ &= -\frac{1}{y^2}\frac{df}{du}.\end{aligned}$$

Consequently

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = -\frac{df}{du} + \frac{df}{du} = 0.$$ 10

(ii) We have that

$$\delta V \approx 2\pi r h \delta r + \pi r^2 \delta h.$$

Hence

$$\begin{aligned}\frac{\delta V}{V} &\approx \frac{2\pi r h \delta r}{V} + \frac{\pi r^2 \delta h}{V} \\ &= \frac{2\delta r}{r} + \frac{\delta h}{h}.\end{aligned}$$

Consequently,

$$\frac{\delta h}{h} \approx -\frac{2\delta r}{r} + \frac{\delta V}{V}.$$

Similar to Q Sheet

Thus

$$\begin{aligned}\left|\frac{\delta h}{h}\right| &\leq \left|\frac{2\delta r}{r}\right| + \left|\frac{\delta V}{V}\right| \\ &= 2 \times 0.004 + 0.006 = 0.014.\end{aligned}$$ 10

IEE (2) 8

8. (i) The equation can be rewritten in the form $x^2 = x + 6$ from which the first scheme follows. Similarly we can write the equation in the form $x(x-1) = 6$ and this leads to the second scheme. Finally, we can write $x = x^2 - 6$ which leads to the third scheme.

The two roots of the equation are $r_1 = 3$ and $r_2 = -2$. The first scheme clearly produces only positive numbers. Furthermore, $|f'(r_1)| < 1$ and hence the first scheme converges to the positive root. For the second scheme we have that $|h'(r_1)| > 1$, $|h'(r_2)| < 1$ which shows convergence to the negative root. For the the third scheme we have that $|g'(r_1)| > 1$, $|g'(r_2)| > 1$ and consequently the scheme does not converge. 10

(ii) Let $f(x) = x^2 + x - \cos x$. We have that $f'(x) = 2x + 1 + \sin x > 0$ in $[0, \pi/2]$ and, hence, the function is strictly increasing. Furthermore, $f(0) = -1 < 0$ and $f(\pi/2) = \pi^2/4 + \pi/2 > 0$. Consequently, there exists a unique root in the interval $[0, \pi/2]$.

We have $f'(x) = 2x + 1 + \sin x$ and consequently the Newton–Raphson scheme is

$$x_{n+1} = x_n - \frac{x_n^2 + x_n - \cos x_n}{2x_n + 1 + \sin x_n}.$$

10

Let now $x_0 = 0.5$. The first iteration gives $x_1 = 0.5514565$. The second iteration gives $x_2 = 0.550010$.

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RBe

MS.

IEE (2) 9

9. (i) The equation is of the form $y'(x) + p(x)y = q(x)$ with $p(x) = 4/x$ and $q(x) = x^4$. The integrating factor is

$$I(x) = e^{\int p(x)\,dx} = e^{\ln x^4} = x^4.$$

Consequently

$$\begin{aligned} y(x) &= cI^{-1}(x) + I^{-1}(x) \int^x I(s)q(s)\,ds \\ &= \frac{c}{x^4} + x^{-4} \int^x s^8\,ds \\ &= \frac{c}{x^4} + \frac{x^5}{9}, \end{aligned}$$

where c is an arbitrary constant. 10

(ii) The equation is of the form

$$N(x,y)\frac{dy}{dx} + M(x,y) = 0$$

with $N(x,y) = x\cos y - 2y$ and $M(x,y) = x + \sin y$. We check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos y,$$

and the equation is exact. The solution of the equation is of the form $z(x, y(x)) = c$ where c is an arbitrary constant and

$$\frac{\partial z}{\partial x} = M \quad \text{and} \quad \frac{\partial z}{\partial y} = N.$$

From the first equation we get that

$$z(x,y) = \frac{1}{2}x^2 + x\sin y + h(y),$$

where $h(y)$ is an arbitrary constant. We differentiate the above formula with respect to y and equate the result to $N(x,y)$ to obtain

$$\frac{\partial z}{\partial y} = x\cos y + h'(y) = x\cos y - 2y.$$

From this we conclude that

$$h(y) = -y^2.$$

Consequently, the solution of the differential equation is defined implicitly through the formula 10

$$\frac{1}{2}x^2 + x\sin y - y^2 = c.$$

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RBall

I EE (2) 10

10. Using the complex version of the Fourier series given in lectures (and in the question)

$$f(x) = c_0 + \sum_{n=1}^{\infty} 2Re(c_n e^{inx})$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ we find

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}\left\{\left[\frac{xe^{-inx}}{-in}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{e^{-inx}}{in}dx\right\}$$

$$= \frac{1}{2\pi}\left[\frac{xe^{-inx}}{-in}\right]_{-\pi}^{\pi} = \frac{i}{2\pi n}\left[\pi e^{-in\pi} + \pi e^{in\pi}\right] = \frac{i}{n}\cos(n\pi) = (-1)^n\frac{i}{n}.$$

Also $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$ and hence

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}Re(ie^{inx}) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx).$$ 10

But the complex form of Parseval's theorem states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2dx = c_0 + 2\sum_{n=1}^{\infty}|c_n|^2 = 2\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4\pi}\int_{-\pi}^{\pi} f(x)^2dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} x^2dx = \frac{\pi^3}{2\pi\cdot 3} = \pi^2/6.$$ 8

Putting $x = -\pi$ into the Fourier series shows that it converges to the value 0, as it also does at $x = +\pi$. This can also be deduced from the form of the jump discontinuity of f at these points. 2