

UNIVERSITY OF LONDON

E1.14 Mathematics 2

B.ENG. AND M.ENG. EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Wednesday 31st May 2006 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

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1. Find the stationary points of the function

$$f(x, y) = (y - 1)(y^2 - x^2)$$

and determine their nature.

Consider the surface $z = f(x, y)$.

Sketch the contour $z = 0$ and other representative contours of the surface.

Indicate the position of the stationary points.

2. (i) If $u = f(x, y)$ and x, y are defined parametrically in terms of s, t via

$$x(s, t) = s^2 + t^2, \quad y(s, t) = 2st,$$

show that

$$s \frac{\partial u}{\partial s} - t \frac{\partial u}{\partial t} = 2(s^2 - t^2) \frac{\partial f}{\partial x}.$$

- (ii) Furthermore show that

$$\frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} = 4(s^2 - t^2) = \left(\frac{\partial s}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial t}{\partial x} \frac{\partial s}{\partial y} \right)^{-1}.$$

PLEASE TURN OVER

3. Show graphically that the equation

$$x^4 - x - 0.1 = 0$$

has one negative solution and one positive solution.

Consider the fixed point schemes

$$(i) x_{n+1} = x_n^4 - 0.1,$$

$$(ii) x_{n+1} = (x_n + 0.1)^{1/4}.$$

Show that (i) with $x_0 = 0$, and (ii) with $x_0 = 1$ are convergent schemes for computing the negative and positive solutions, respectively.

Compute the first two iterates for both schemes.

Write down the Newton-Raphson scheme for computing these solutions.

Starting with the same x_0 as above, compute the first two Newton-Raphson iterates for both solutions.

4. Two planes are defined by the equations

$$\begin{aligned} x - 4y + z &= 6, \\ x - y + z &= 2. \end{aligned}$$

- (i) Find the perpendicular distance from the origin to each plane.
- (ii) Find the vector equation of the straight line defined by the intersection of these planes.

5. Consider the $n \times n$ matrix

$$A = I - B.$$

where $B^m = 0$ for some positive integer m .

Show that

$$A^{-1} = I + B + B^2 + \dots + B^{m-1}.$$

Given

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

show that $B^3 = 0$.

Hence find A^{-1} . Use the Binomial Theorem to find A^p where $p \geq 2$ is an integer.

Calculate A^2 by direct multiplication and check your result.

6. (i) Using the substitution $v = y/x$, or otherwise, find an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{x}{2x - y}$$

that satisfies $y = 0$ at $x = 1$.

Give an expression for x in terms of y/x .

(ii) Find the general solution $y(x)$ of the differential equation

$$(x^2 + 1) \frac{dy}{dx} + xy = (x^2 + 1)^{1/2}.$$

Evaluate the arbitrary constant given that

$$\lim_{x \rightarrow \infty} x(y - 1) = 1.$$

PLEASE TURN OVER

7. (i) Consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{xy}{(x^2 + y^2)}$$

and, using the substitution $y = ux$, show that y and x are related via

$$x^2 = 2y^2 \ln[Ay],$$

where A is an arbitrary constant.

- (ii) Solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}.$$

8. (i) Find the solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 20 \sin 2x,$$

that satisfies $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

- (ii) Find the general solution $y(x)$ of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = e^{4x}.$$

9. Consider the differential equation

$$(1+x^2) \frac{d^2y}{dx^2} + (2x-1) \frac{dy}{dx} = 0. \quad (1)$$

Use Leibnitz's theorem to show by differentiating n times that

$$(1+x^2) y^{(n+2)}(x) + [2(n+1)x-1] y^{(n+1)}(x) + n(n+1) y^{(n)}(x) = 0.$$

Hence show that, if $y(0) = y'(0) = 1$, the Taylor-MacLaurin expansion of $y(x)$ is

$$y(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$$

Show also that the substitution if $v = y'$ reduces equation (1) to a first order equation and hence find a closed form solution for $y(x)$ subject to the same conditions

$$y(0) = y'(0) = 1.$$

10. Given the function $f(x) = \cos x$ for $0 < x \leq \pi$ and $f(x) = -\cos x$ in $-\pi < x \leq 0$, show that

$$\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2-1} \sin 2mx$$

holds in $0 < x \leq \pi$.

Hence show that

$$\frac{1}{\sqrt{2}} = \frac{8}{\pi} \sum_{p=1}^{\infty} \frac{(2p-1)(-1)^{p+1}}{16p^2-16p+3}.$$

END OF PAPER

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^{n+1}}{(n+1)} + \dots \quad (-1 < x \leq 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b;$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

$$\cos iz = \cosh z; \quad \cosh iz = \cos z; \quad \sin iz = i \sinh z; \quad \sinh iz = i \sin z.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f D^n g + \binom{n}{1} Df D^{n-1} g + \dots + \binom{n}{n} D^n f D^0 g + \dots + D^n f g.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^{(n)}(a) + \epsilon_n(h),$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$, $0 < \theta < 1$.

(c) Taylor's expansion of $f(x, y)$ about (a, b) :

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + \frac{1}{2!} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

- i. If $y = y(x)$, then $f = F(x)$, and $\frac{dF}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.
- ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.
- iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and $\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}$, $\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}$.

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.

Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.

If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;

If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;

If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\frac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$
 $\sin \theta = 2t/(1+t^2)$, $\cos \theta = (1-t^2)/(1+t^2)$, $d\theta = 2dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (\alpha^2 - x^2)^{-1/2} dx = \sin^{-1} \left(\frac{x}{\alpha} \right), \quad |x| < \alpha.$$

$$\int (\alpha^2 + x^2)^{-1/2} dx = \sinh^{-1} \left(\frac{x}{\alpha} \right) = \ln \left\{ \frac{x}{\alpha} + \left(1 + \frac{x^2}{\alpha^2} \right)^{1/2} \right\}.$$

$$\int (\alpha^2 - \alpha^2)^{-1/2} dx = \cosh^{-1} \left(\frac{x}{\alpha} \right) = \ln \left[\frac{x}{\alpha} + \left(\frac{x^2}{\alpha^2} - 1 \right)^{1/2} \right].$$

$$\int (\alpha^2 + x^2)^{-1} dx = \left(\frac{1}{\alpha} \right) \tan^{-1} \left(\frac{x}{\alpha} \right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and $x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2, \dots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x) dx \approx (h/2) [y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x) dx \approx (h/3) [y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x) dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	$d^2 f/dt^2$	$s^2 F(s) - sf(0) - f'(0)$
$e^{at} f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial \alpha) f(t, \alpha)$	$(\partial/\partial \alpha) F(s, \alpha)$	$\int_0^t f(t) dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2, \dots)$	$n!/s^{n+1}, (s > 0)$
e^{at}	$1/(s-a), (s > a)$	$\sin \omega t$	$\omega/(s^2 + \omega^2), (s > 0)$
$\cos \omega t$	$s/(s^2 + \omega^2), (s > 0)$	$H(t-T)$	$e^{-sT}/s, (s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots, \text{ and}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots$$

Parseval's theorem

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EEI(2) Q1P1
Question SOLN		Marks & seen/unseen
Parts	$\frac{\partial f}{\partial x} = (y-1)(-2x) \quad \text{--- (1)}$ $\frac{\partial f}{\partial y} = y^2 - x^2 + 2y(y-1) \quad \text{--- (2)}$ <p>For stationary point we need</p> $\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0.$ <p>From (1) $x = 0$ or $y = 1$</p> <p>From (2) $y = 0$ or $\frac{2}{3}$ $x = \pm 1$</p> <p>Thus 4 st. pts</p> <p>$P_1(0,0), P_2(0, \frac{2}{3}), P_3(1,1), P_4(-1,1)$</p> <p>Nature determined by</p> $\Delta = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y} \right]^2$ $\frac{\partial^2 f}{\partial x^2} = -2(y-1), \quad \frac{\partial^2 f}{\partial y^2} = 6y-2$ $\frac{\partial^2 f}{\partial x \partial y} = -2x$	<p style="text-align: right;">2</p> <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> <p style="text-align: right;">4</p> <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> <p style="text-align: right;">2</p> <hr style="width: 50%; margin-left: auto; margin-right: 0;"/> <p style="text-align: right;">2</p>
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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course
Q1 p2

Question
SOLN

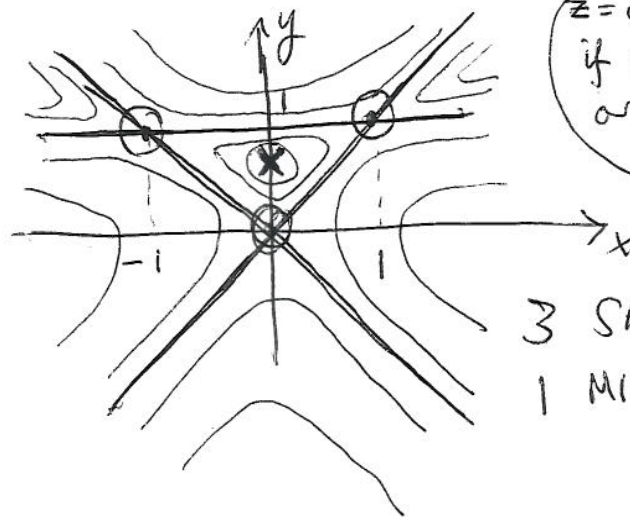
Marks &
seen/unseen

Parts

	P_1	P_2	P_3	P_4
$\frac{\partial^2 f}{\partial x^2}$	+2	$\frac{2}{3}$	0	0
$\frac{\partial^2 f}{\partial y^2}$	-2	2	4	+8
$\frac{\partial^2 f}{\partial x \partial y}$	0	0	-2	+2
Δ	-4	$\frac{4}{3}$	-4	-4
$z=f(x,y)$	0	$-\frac{4}{27}$	0	0
NATURE	SADDLE	MINIMUM	SADDLE	SADDLE

Formula
sheet

4



$z=0$
if $x=\pm y$
or $y=1$

2

3 SADDLES AT O
1 MINIMUM AT X

4

TOTAL 20

Setter's initials
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Checker's initials
RB

Page number

Solution 1

(a) $x = s^2 + t^2$

$y = 2st$

$u = f$

$u_s = f_x x_s + f_y y_s = 2s f_x + 2t f_y \quad xt$

$u_t = f_x x_t + f_y y_t = 2t f_x + 2s f_y \quad xs. \text{ and subtract.}$

$5u_s - tu_t = 2(s^2 - t^2) f_x.$

6 marks

(b)

$x_s = 2s$

$x_t = 2t$

$y_s = 2t$

$y_t = 2s$

$x_s y_t - x_t y_s = 2s \cdot 2s - 4t^2 = 4(s^2 - t^2).$

4 marks

$x+y = (s+t)^2$

$x-y = (s-t)^2$

$2(s+t)[s_x + t_x] = 1$

$2(s-t)[s_x - t_x] = 1.$

$s_x = \frac{1}{4(s+t)} + \frac{1}{4(s-t)} = \frac{1 \cdot s}{2(s^2 - t^2)} = s_y \quad \text{by interchanging variables.}$

$t_y = \frac{1}{4(s+t)} - \frac{1}{4(s-t)} = \frac{-t}{2(s^2 - t^2)} = s_y$

5 marks

$\therefore s_x t_y - t_x s_y = \frac{1}{4(s^2 - t^2)} \quad \text{as reqd}$

5 marks

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

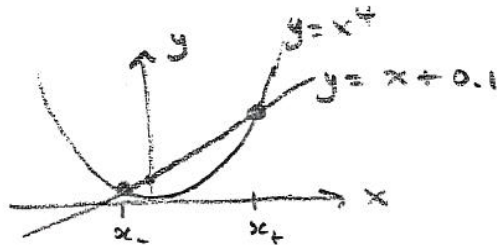
Course 03

Question

Q7 Solution

Marks &
seen/unseen

Parts



$$x^4 = x + 0.1$$

$$-0.1 < x_- < 0$$

$$1 < x_+ < 1.1$$

3

$$x_{n+1} = g(x_n)$$

$$(i) \quad g(x) = x^4 - 0.1 \Rightarrow g'(x) = 4x^3$$

$$|g'(x)| < 1 \quad \text{if} \quad |x| < \left(\frac{1}{4}\right)^{1/3}$$

$$\therefore x_0 = 0, \text{ convergent to } x_- \text{ as } |x_-| < \left(\frac{1}{4}\right)^{1/3}$$

3

$$(ii) \quad g(x) = (x + 0.1)^{1/4} \Rightarrow g'(x) = \frac{1}{4}(x + 0.1)^{-3/4}$$

$$|g'(x)| < 1 \quad \text{if} \quad |x + 0.1| \geq \left(\frac{1}{4}\right)^{4/3}$$

$$\therefore x_0 = 1, \text{ convergent to } x_+ > 1$$

3

$$(i) \quad x_0 = 0, \quad x_1 = -0.1, \quad x_2 = -0.0999$$

2

$$(ii) \quad x_0 = 1, \quad x_1 = (1.1)^{1/4} \approx 1.0241$$

$$x_2 \approx (1.1241)^{1/4} \approx 1.0297$$

2

$$f(x) = x^4 - x - 0.1 \Rightarrow f'(x) = 4x^3 - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{3x_n^4 + 0.1}{4x_n^3 - 1}$$

3

$$x_0 = 0, \quad x_1 = -0.1, \quad x_2 = \frac{-0.1003}{1.004} \approx -0.0999$$

2

$$x_0 = 1, \quad x_1 = \frac{3.1}{3} = 1.0333, \quad x_2 \approx 1.0313$$

2

/20

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EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course

EE1P2

Question
SOLN

Q4

Marks &
seen/unseen

Parts

(i)

$x - 4y + z = 6$
can be written

$\underline{r} \cdot \underline{n}_1 = 6$
with $\underline{n}_1 = (1, -4, 1)$

Normalize

$$\underline{r} \cdot \hat{\underline{n}}_1 = 6/\sqrt{18} \quad \text{--- (1)}$$

$$\hat{\underline{n}}_1 = \frac{1}{\sqrt{18}} (1, -4, 1) \leftarrow \underline{\text{unit normal}}$$

Similarly $x - y + z = 2$

can be written

$$\underline{r} \cdot \hat{\underline{n}}_2 = 2/\sqrt{3} \quad \text{--- (2)}$$

$$\hat{\underline{n}}_2 = \frac{1}{\sqrt{3}} (1, -1, 1) \leftarrow \underline{\text{unit normal}}$$

Perp. dist from 0 to plane 1 is
r.h.s. of eqn. (1) i.e. $6/\sqrt{18} = \sqrt{2}$

Perp. dist. from 0 to plane 2 is
r.h.s. of eqn. (2) i.e. $2/\sqrt{3}$

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Page number

Question
SOLNⓄ
Marks &
seen/unseen

Parts

(ii)

Vector equation of line is

$$\underline{r} = \underline{a} + \lambda \underline{d}$$

where \underline{a} is a point in plane and
 \underline{d} is direction vector.

\underline{d} is perp. to both \underline{n}_1 and \underline{n}_2

$$\therefore \underline{d} = \underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -4 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

← determinant
not
needed
for this
computation

$$= \underline{i}(-3) - \underline{j}(0) + \underline{k}(3)$$

$$= (-3, 0, 3)$$

Choose \underline{a} to be any point in both
planes. One choice is $z=0$
 $x-4y=6$ and $x-y=2$

$$\text{i.e. } y = -4/3, \quad x = 2/3$$

$$\text{i.e. } \underline{r} = (2/3, -4/3, 0) + \lambda (-3, 0, 3)$$

Other valid choices possible

2
Total 20

Setter's initials

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Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course

EE1

p2

Question

SOLN

Marks & 05
seen/unseen

Parts

The following multiplication verifies that A^{-1} is of the given form

$$\begin{aligned}
 & (I - B)(I + B + B^2 + \dots + B^{m-1}) \\
 &= I - B + B - B^2 + B^2 - B^3 + \dots + B^{m-1} - B^m \\
 &= I \quad (\text{cancellation in pairs} + B^m = 0)
 \end{aligned}$$

4

$$B^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 2$$

$$B^3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 2$$

Thus $A^{-1} = I + B + B^2$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

2

Setter's initials

RLJ

Checker's initials

RB

Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course
EE1 P2

Question

SOLN

Marks &
seen/unseen

Parts

$$A^p = (I - B)^p = I - pB + \frac{p(p-1)}{1 \cdot 2} B^2$$

from binomial theorem and fact
that $B^3 = 0$

5

$$A^2 = I - 2B + B^2$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & +2 \\ 0 & 0 & 1 \end{pmatrix}$$

3

Direct multiplication

$$A^2 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

2

CHECKS

Total

(20)

Setter's initials

RLJ

Checker's initials

RB

Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06		Course EE1 P2
Question SOLN		Q6 Marks & seen/unseen
Parts	<p>(i) Substitute $y = xv(x)$ in given equation to get</p> $v + x \frac{dv}{dx} = \frac{1}{2-v}$ <p>Thus</p> $x \frac{dv}{dx} = \frac{1-2v+v^2}{2-v}$ <p>and on separation</p> $\int \frac{dx}{x} = \int \frac{2-v}{(1-v)^2} dv$ <p>ie.</p> $\ln x = -\ln 1-v + \frac{1}{1-v} + C$ <p>$y=0$ when $x=1 \Rightarrow v=0$ and $\therefore C=-1$</p> <p>Hence</p> $\ln x = -\ln\left 1-\frac{y}{x}\right + \frac{x}{y-x} - 1,$ <p>Or equivalent form.</p> <p>No simplification needed.</p> <p><u>NB. 1 mark lost if modulus signs omitted.</u></p>	<p>2</p> <hr/> <p>2</p> <hr/> <p>2</p> <hr/> <p>2</p> <hr/>
Setter's initials RLJ	Checker's initials RB	Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course

EE1 p2

Question

Soln

Q6

Marks &
seen/unseen

Parts

(ii)

Rewrite in standard form

$$\frac{dy}{dx} + \frac{x}{x^2+1} y = (x^2+1)^{-1/2}$$

$$\text{I.F.} = \exp \left[\int \frac{x}{x^2+1} dx \right]$$

$$= \exp \left[\frac{1}{2} \ln(x^2+1) \right] = (x^2+1)^{1/2}$$

$$\therefore (x^2+1)^{1/2} \frac{dy}{dx} + x(x^2+1)^{-1/2} y = 1$$

$$\therefore \frac{d}{dx} \left[(x^2+1)^{1/2} y \right] = 1$$

$$\therefore (x^2+1)^{1/2} y = x + C \checkmark$$

$$\text{or } y = \frac{x}{(x^2+1)^{1/2}} + \frac{C}{(x^2+1)^{1/2}}$$

Now consider

$$\lim_{x \rightarrow \infty} x(y-1) = \lim_{x \rightarrow \infty} \left\{ x \left[\frac{x}{(x^2+1)^{1/2}} - 1 \right] + \frac{cx}{(x^2+1)^{1/2}} \right\}$$

1st term on r.h.s. $\rightarrow 0$ as can be seen
from binomial th. or otherwise

Setter's initials

RL

Checker's initials

RB

Page number

1

2

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2

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course
EE1P2
Q6

Question

SOLN

Marks &
seen/unseen

Parts

$$\text{eg } x \left[\frac{x}{(x^2+1)^{1/2}} - 1 \right] = x \left[\left(1 + \frac{1}{x^2} \right)^{-1/2} - 1 \right]$$

$$= x \left[1 - \frac{1}{2} \frac{1}{x^2} + O\left(\frac{1}{x^4}\right) - 1 \right]$$

$$\rightarrow 0$$

2nd term on right $\rightarrow C$

$$\therefore C = 1 \checkmark$$

$$\text{and solution is } y = \frac{x}{(x^2+1)^{1/2}} + \frac{1}{(x^2+1)^{1/2}} + 1$$

3

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Total 20

Setter's initials

RLJ

Checker's initials

RB

Page number

2.

$$(a) \quad \frac{dy}{dx} = \frac{xy}{x^2+y^2} = \frac{1}{\frac{x}{y}} + \frac{y}{x}$$

$$\text{let } y = ux$$

$$x \frac{du}{dx} + u = \frac{1}{u + \frac{1}{u}}$$

$$x \frac{du}{dx} = \frac{u}{u^2+1} - u = -\frac{u^3}{u^2+1}$$

$$\frac{dx}{x} = -\frac{(u^2+1)du}{u^3} = \left(-\frac{1}{u} - \frac{1}{u^3}\right) du$$

3 marks

$$\log x = -\log u + \frac{1}{2u^2} + \text{Const.}$$

$$\log(xu) = \frac{1}{2u^2} + \text{Constant.}$$

$$\log y = \frac{1}{2\left(\frac{y^2}{x^2}\right)} + \log A$$

$$2y^2 \log Ay = x^2$$

7 marks.

$$(b) \quad \frac{dy}{dx} = \frac{x+y+1}{x+y-1}$$

$$\text{let } u = x+y$$

$$\frac{du}{dx} - 1 = \frac{dy}{dx}$$

3 marks

$$\frac{du}{dx} - 1 = \frac{u+1}{u-1} \Rightarrow \frac{du}{dx} = \frac{2u}{u-1} \quad 2dx = \left(1 - \frac{1}{u}\right) du$$

$$2x+c = u - \log u$$

$$2x+c = x+y - \log(x+y)$$

$$\cancel{y} + y - x + \text{const} = \log(x+y) \quad 7 \text{ marks.}$$

R.3

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course
EE1 P2
Q8

Question
SOLN

Marks &
seen/unseen

Parts

(i)

Try complementary function $y = e^{\lambda x}$

This gives auxiliary equation

$$\lambda^2 - 4\lambda + 8 = 0$$

$$\text{i.e. } \lambda = 2 \pm \sqrt{4-8} = 2 \pm 2i$$

So complementary function is

$$y = (A \sin 2x + B \cos 2x) e^{2x}$$

For particular integral try

$$y = C \sin 2x + D \cos 2x$$

Then equation gives

$$\begin{aligned} & [-4C + 8D + 8C] \sin 2x \\ & + [-4D - 8C + 8D] \cos 2x = 20 \sin 2x \end{aligned}$$

$$\text{i.e. } +4C + 8D = 20$$

$$-8C + 4D = 0$$

$$\Rightarrow D = 2, C = 1$$

Hence general solution is

$$y = (A \sin 2x + B \cos 2x) e^{2x} + \sin 2x + 2 \cos 2x$$

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Page number

EXAMINATION QUESTIONS/SOLUTIONS 2005-06

Course
EE1 P2

Question
SOLN

Q 8
Marks &
seen/unseen

Parts

$$y' = 2(A \sin 2x + B \cos 2x)e^{2x} + (2A \cos 2x - 2B \sin 2x)e^{2x} + 2 \cos 2x - 4 \sin 2x$$

Conditions $\Rightarrow B + 2 = 0$
and $2B + 2A + 2 = 0$

i.e. $B = -2, A = 1$

and $y = (\sin 2x - 2 \cos 2x)e^{2x} + \sin 2x + 2 \cos 2x$

1

3

(ii)

Try complementary function $y = e^{\lambda x}$,
gives aux. eqn. $\lambda^2 - 5\lambda + 4 = 0$
 $\lambda = 4$ or 1

Complementary function is then

$$y = A e^{4x} + B e^x$$

Note inhomogeneous term, e^{4x} , appears
in complementary function. Therefore
try $y = Cx e^{4x}$ for particular integral

2

2

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Setter's initials
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Checker's initials
43

Page number

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EPE102 Q8
Question SOLN		Marks & seen/unseen
Parts	<p>This gives $y' = Ce^{4x} + 4Cx e^{4x}$ $y'' = 8Ce^{4x} + 16Cx e^{4x}$ and subst. in given eqn. we get</p> $8Ce^{4x} + 16Cx e^{4x} - 5Ce^{4x} - 20Cx e^{4x} + 4Cx e^{4x} = e^{4x}$ <p>Collecting terms</p> $(8C - 5C - 1)e^{4x} + (16C - 20C + 4C)xe^{4x} = 0$ <p>i.e. $3C - 1 = 0 \quad C = \frac{1}{3}$</p> <p>and general solution is</p> $y = Ae^{4x} + Be^x + \frac{1}{3}xe^{4x}$	<p>2</p> <hr/> <p>2</p> <hr/> <p>1</p> <hr/> <p>Total 20</p>
Setter's initials	Checker's initials	Page number
RLJ	RB	

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EE1 P2 ④9
Question SOLN		Marks & seen/unseen
Parts	<p>Differentiating n times by Leibnitz</p> $(1+x^2) y^{(n+2)}(x) + n \cdot 2x y^{(n+1)}(x) + \frac{2n(n-1)}{2} y^{(n)}(x)$ $+ (2x-1) y^{(n+1)}(x) + n \cdot 2 y^{(n)}(x) = 0$ <p>Thus</p> $(1+x^2) y^{(n+2)}(x) + [2(n+1)x-1] y^{(n+1)}(x)$ $+ n(n+1) y^{(n)}(x) = 0 \quad \checkmark$ <p>Now put $x=0$ in this to get</p> $y^{(n+2)}(0) - y^{(n+1)}(0) + n(n+1) y^{(n)}(0) = 0$ <p>This is a recurrence relation which determines $y^{(n)}(0)$ for $n \geq 2$ given $y(0)$ and $y^{(1)}(0)$. Thus</p> $y(0) = 1$ $y^{(1)}(0) = 1$ $n=0 \quad y^{(2)}(0) = y^{(1)}(0) = 1 \quad \checkmark$ $n=1 \quad y^{(3)}(0) = y^{(2)}(0) - 2y^{(1)}(0) = -1$ <p>Enough!</p>	<p>2</p> <hr/> <p>2</p> <hr/> <p>2</p> <hr/> <p>1</p> <hr/> <p>3</p> <hr/>
Setter's initials	Checker's initials	Page number
RLJ	RB	

	EXAMINATION QUESTIONS/SOLUTIONS 2005-06	Course EEEL(2) Q9
Question		Marks & seen/unseen
Parts	<p>Taylor-MacLaurin expansion is</p> $y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \dots$ $= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$	2
	<p>Substitution gives</p> $(1+x^2) \frac{dv}{dx} + (2x-1)v = 0$ <p>Separate $\int \frac{dv}{v} = -\int \frac{2x-1}{1+x^2} dx$</p> $\ln v - 1 = -\ln(1+x^2) + \tan^{-1}x + C$ <p>$y' = 1$ when $x=0 \Rightarrow v=1 \Rightarrow C=0$</p> <p>So $\frac{dy}{dx} = v = \frac{1}{1+x^2} \exp(\tan^{-1}x)$</p> <p>Integrate $\Rightarrow y = \exp(\tan^{-1}x) \checkmark$</p>	<p>2</p> <hr/> <p>3</p> <hr/> <p>2</p> <hr/> <p>1</p>
Setter's initials	Checker's initials	Page number

RL

RB

$$3) f(x) = \begin{cases} \cos x & 0 < x \leq \pi \\ -\cos x & -\pi < x \leq 0 \end{cases}$$

Use a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

3 marks

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \, dx \\ &= -\frac{1}{\pi} \left\{ \frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right\} \Big|_0^{\pi} \\ &= -\frac{1}{\pi} \left\{ \frac{(-1)^{n+1} - 1}{(n+1)} + \frac{(-1)^{n-1} - 1}{(n-1)} \right\} \end{aligned}$$

3 marks

$$= 0 \quad n \text{ odd}$$

$$= \frac{2}{\pi} \left(\frac{2}{(n+1)} + \frac{2}{(n-1)} \right) = \frac{4n}{\pi(n^2-1)}$$

$$\therefore f = \sum_{m=1}^{\infty} \frac{4 \cdot 2m}{\pi \cdot \pi(2m)^2 - 1} \sin 2mx$$

4 marks

$$\text{Set } x = \frac{\pi}{4} \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

3 marks

$$\therefore \frac{1}{\sqrt{2}} = \sum_{m=1}^{\infty} \frac{8m}{\pi[4m^2-1]} \sin \left[\frac{m\pi}{2} \right]$$

" 0 if m even

Set $m = 2p - 1$ so odd for integer p .

$\begin{cases} p=1,3,\dots = + \\ p=2,\dots = - \end{cases}$

$$= \sum_{p=1}^{\infty} \frac{8[2p-1]}{\pi[4(4p^2+1-4p)-1]} \sin \left[\frac{(2p-1)\pi}{2} \right] = (-1)^{p+1}$$

$$\frac{1}{\sqrt{2}} = \sum_{p=1}^{\infty} \frac{8(2p-1)}{\pi[16p^2+3-16p]} (-1)^{p+1}$$

7 marks

R3