(EE - stream)

(E1.14)

UNIVERSITY OF LONDON

[I(2)E 2002]

B.ENG. AND M.ENG. EXAMINATIONS 2002

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART I : MATHEMATICS 2 (ELECTRICAL ENGINEERING)

Thursday 30th May 2002 10.00 am - 1.00 pm

Answer EIGHT questions.

[Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. Show that the four stationary points of the function

$$z(x,\, y) \;=\; (y - x)(2x^2 + y^2 - 3)$$

lie either on the line $y = x$ or on the line $y = -2x$, and determine their nature.

Sketch the contours through the saddle-points and some general contours of $z(x,\, y)$.

Indicate the position of the stationary points on your sketch.

2. Show that $u(x,\, y) \;=\; \tan^{-1}\left(\frac{y}{x}\right)$ and $v(x,\, y) \;=\; \tan^{-1}\left(\frac{x}{y}\right)$ both satisfy the equation

$$x\,\frac{\partial w}{\partial x} \;+\; y\,\frac{\partial w}{\partial y} \;=\; 0.$$

Hence, show that

$$z(x,\, y) \;=\; x^2 u \;-\; y^2 v$$

satisfies the equation

$$x\,\frac{\partial z}{\partial x} \;+\; y\,\frac{\partial z}{\partial y} \;=\; 2z\;.$$

Show also that

$$\frac{\partial^2 z}{\partial x^2} = 2u \;+\; 2x\,\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} \;=\; -\,2v - 2y\,\frac{\partial v}{\partial y}\;.$$

PLEASE TURN OVER

3. A numerical approximation of $I = \int_0^{0.8} f(x)\,dx$, where $f(x) = \sqrt{1+x}$, is given by the trapezium rule with two intervals as 0.9416 (correct to 4 decimal places).

Find further approximations by :

(i) using the Trapezium rule with four intervals;

(ii) using Richardson's extrapolation;

(iii) expanding $f(x)$ using the binomial theorem in a series up to and including the term proportional to x^2 and integrating the terms of the resulting series.

Calculate I exactly and compare with the most accurate approximation.

All calculations should be rounded off to four decimal places.

Richardson Extrapolation: Let $I = \int_a^b f(x)dx$ and let I_1 and I_2 be two estimates of I obtained using the Trapezium rule with intervals h and $h/2$. Then provided h is small enough $(4I_2 - I_1)/3$ is a better estimate of I.

4. (i) Show that, for any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) \;=\; 2\mathbf{b} \times \mathbf{a}.$$

(ii) Show that, for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$$\{(\mathbf{a}+\mathbf{b}) \times (\mathbf{b}+\mathbf{c})\} \bullet (\mathbf{c}+\mathbf{a}) = 2(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}.$$

Verify this result for the special case where $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (1, 1, 0)$ and $\mathbf{c} = (1, 1, 1)$.

(iii) The vector $\mathbf{x}$ satisfies the pair of equations

$$\mathbf{a} \times \mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{a} \bullet \mathbf{x} = 3,$$

where $\mathbf{a} = (1, 2, 1)$ and $\mathbf{b} = (7, -1, -5)$.

By taking the vector product of the first equation with $\mathbf{a}$, or otherwise, determine $\mathbf{x}$.

5. (i) Find the minimum distance from the origin to the plane P given by the equation

$$x \; - \; 2y \; + \; 3z \; = \; 14.$$

(ii) Another plane Q has equation $x - \alpha y = 0$. Find the value of α so that P and Q are orthogonal.

(iii) For this value of α, let l be the straight line which is the intersection of these two planes. Find an equation for l in the form $\mathbf{r}(\lambda) = \lambda\mathbf{a} + \mathbf{b}$.

6. Let

$$A \; = \; \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} .$$

Find the value of a such that $A^3 = I$.

Find the value of a such that $A^4 = I$.

For each of these two values of a find A^{-1}.

Prove that A^{-1} exists for any a, but that there is no value of a such that $A = A^{-1}$.

7. Using Gaussian Elimination, or otherwise, find the values of the constants λ and μ for which the equations

$$\begin{array}{rcrcrcl} x & - & y & + & 3z & = & 1, \\ 2x & + & 3y & + & \lambda z & = & 7, \\ x & + & y & + & 2z & = & \mu \end{array}$$

have infinitely many solutions, and find these solutions.

If $\lambda \; = \; \frac{7}{2} + 5\alpha$ and $\mu = 3 - 2\beta$, find the solution of the above equations in terms of the non-zero constants α and β.

PLEASE TURN OVER

8. (i) Find the solution of the differential equation

$$x \frac{dy}{dx} + (1+x)\,y = x ,$$

subject to the condition that $y = 1$ when $x = 2$.

(ii) The function $y(x)$ satisfies the differential equation

$$x \frac{dy}{dx} = y + \frac{y^2}{1+x^2} ,$$

subject to the condition that $y = 4/\pi$ when $x = 1$.

Using the substitution $v = y/x$, or otherwise, solve for $y(x)$.

9. (i) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 8 + e^{-x} .$$

(ii) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = \cos 2x .$$

10. The function $f_1(x)$ is periodic, with period 2π, and is an odd function of x. In the interval $0 < x < \pi$ it has the value

$$f_1(x) = \pi - x, \qquad 0 < x < \pi.$$

Sketch the graph of $f_1(x)$ over the interval $-2\pi < x < 2\pi$.

Find the Fourier series for $f_1(x)$. State the values of the Fourier series when $x = 0$ and when $x = \pi/2$. Use the latter result to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} .$$

The function $f_2(x)$ also has period 2π and is an even function. In the interval $0 < x < \pi$, $f_2(x)$ is defined to be equal to $f_1(x)$. Sketch the graph of $f_2(x)$ over the interval $-2\pi < x < 2\pi$ and find its Fourier series.

END OF PAPER

First yr Data sheet

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \dots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n\frac{x^{n+1}}{(n+1)} + \dots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1}Df\,D^{n-1}g + \dots + \tbinom{n}{r}D^r f\,D^{n-r}g + \dots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \dots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!,\quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \dots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx)]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a .$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\} .$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right| .$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right) .$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15 ,$$

is a better estimate of I .

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s , $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} , \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots , \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) .$$

September 2000

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

(EE-stream) E1

PAPER I(2)

QUESTION

SOLUTION 1

Please write on this side only, legibly and neatly, between the margins

$z(x,y) = (y-x)(2x^2+y^2-3)$

$z_x = -(2x^2+y^2-3)+4x(y-x) = 4xy-6x^2-y^2+3$

$z_y = (2x^2+y^2-3)+2y(y-x) = -2xy+2x^2+3y^2-3$ 2

Stationary points: $z_x=0,\ z_y=0$.

Adding $\to 2xy-4x^2+2y^2=0 \to y^2+xy-2x^2=0$

$\therefore (y+2x)(y-x)=0 \quad \therefore y=x,\ -2x$.

(i) $y=x$: $z_x = 4x^2-6x^2+x^2+3=0 \quad \therefore x=\pm 1$

(ii) $y=-2x$: $z_x = -8x^2-6x^2-4x^2+3=0 \quad \therefore x = \pm\frac{1}{\sqrt6}$

$\therefore$ 4 stat^y pts: $(1,1), (-1,-1), \left(\frac{1}{\sqrt6},-\frac{2}{\sqrt6}\right), \left(-\frac{1}{\sqrt6},\frac{2}{\sqrt6}\right)$ 4

Nature: $z_{xx}=4y-12x,\ z_{yy}=-2x+6y,\ z_{xy}=4x-2y$

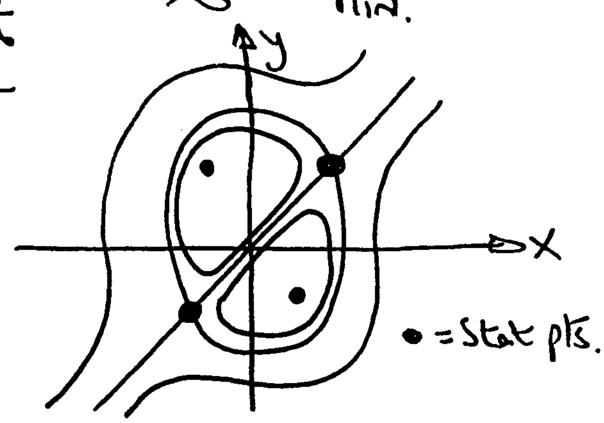
	z_{xx}	z_{yy}	z_{xy}	$z_{xx}z_{yy}-z_{xy}^2$	
$(1,1)$	-8	4	2	$<$	Saddle pt: $z(1,1)=0$
$(-1,-1)$	8	-4	-2	$<$	" : $z(-1,-1)=0$
$\left(\frac{1}{\sqrt6},-\frac{2}{\sqrt6}\right)$	$-\frac{20}{\sqrt6}$	$-\frac{14}{\sqrt6}$	$\frac{8}{\sqrt6}$	>0	Max.
$\left(-\frac{1}{\sqrt6},\frac{2}{\sqrt6}\right)$	$\frac{20}{\sqrt6}$	$\frac{14}{\sqrt6}$	$-\frac{8}{\sqrt6}$	>0	Min.

4

Contours through saddle-points: $z=0$

ie $y=x$,

$2x^2+y^2=3$



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Setter :

Setter's signature :

Checker : FENNER

Checker's signature : R. Y. Fenner

$\tan u = \frac{y}{x}$ $\therefore \sec^2 u \, u_x = -\frac{y}{x^2}$ $\therefore \left(1+\frac{y^2}{x^2}\right) u_x = -\frac{y}{x^2}$

$\therefore u_x = -\frac{y}{(x^2+y^2)}$. $\sec^2 u . u_y = \frac{1}{x}$

$\therefore u_y = \frac{x}{(x^2+y^2)}$. $\therefore x u_x + y u_y = 0$. 3

$\tan v = \frac{x}{y}$ $\therefore v_x = \frac{y}{(x^2+y^2)}$ and $v_y = -\frac{x}{(x^2+y^2)}$

$\therefore x v_x + y v_y = 0$ 3

$z_x = 2xu - y^2 v_x + x^2 u_x = 2xu - \frac{y^3}{(x^2+y^2)} - \frac{x^2 y}{(x^2+y^2)}$

$\therefore z_x = 2xu - y$. 3

$z_y = x^2 u_y - 2yv - y^2 v_y = -2yv + \frac{x^3}{(x^2+y^2)} + \frac{y^2 x}{(x^2+y^2)}$

$\therefore z_y = -2yv + x$ 3

$\therefore x z_x + y z_y = 2x^2 u - xy - 2y^2 v + xy = 2z$. 1

$z_x = 2xu - y$ $\therefore z_{xx} = 2u + 2x u_x$ 1

$z_y = -2yv + x$ $\therefore z_{yy} = -2v - 2y v_y$ 1

Setter : DRCH Setter's signature : ARChadwick

Checker : WILSON Checker's signature : J. Wilson

15

(i) $I_2 = \frac{0.2}{2}\left\{1+2\left(\sqrt{1.2}+\sqrt{1.4}+\sqrt{1.6}\right)+\sqrt{1.8}\right\} = 0.9429$ 3

(ii) $I \simeq \frac{4I_2 - I_1}{3} = \frac{4\times 0.9429 - 0.9416}{3} = 0.9433$ 4

(iii) $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}-1\right)\frac{x^2}{2!} = 1+\frac{x}{2}-\frac{x^2}{8}$

$$I \simeq \int_0^{0.8}\left(1+\frac{x}{2}-\frac{x^2}{8}\right)dx = \left[x+\frac{x^2}{4}-\frac{x^3}{24}\right]_0^{0.8} = 0.9387$$ 5

Exact answer:

$$I = \int(1+x)^{0.5}dx = \left[\frac{(1+x)^{1.5}}{1.5}\right]_0^{0.8} = \frac{1}{1.5}\left(1.8^{1.5}-1\right)$$

$$= 0.9433$$

Therefore approximation using Richardson's extrapolation is exact to 4 decimal places accuracy. 3

15

Please write on this side only, legibly and neatly, between the margins

(i) $(\underline{a}+\underline{b})\times(\underline{a}-\underline{b}) = \underline{a}\times\underline{a} - \underline{a}\times\underline{b} + \underline{b}\times\underline{a} - \underline{b}\times\underline{b}$

Use $\underline{a}\times\underline{a}=0$, $\underline{b}\times\underline{b}=0$, $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$ to get

$\underline{(\underline{a}+\underline{b})\times(\underline{a}-\underline{b}) = 2\,\underline{b}\times\underline{a}}$ 4

(ii) $(\underline{a}+\underline{b})\times(\underline{b}+\underline{c}) = \underline{a}\times\underline{b} + \cancel{\underline{b}\times\underline{b}}^{\,0} + \underline{a}\times\underline{c} + \underline{b}\times\underline{c}$

so LHS of identity is $(\underline{a}\times\underline{b} + \underline{a}\times\underline{c} + \underline{b}\times\underline{c})\cdot(\underline{c}+\underline{a}) = (\underline{a}\times\underline{b})\cdot\underline{c} + \underline{a}\cdot(\underline{b}\times\underline{c})$

Using $\underline{a}\times\underline{b}\cdot\underline{a} = 0$ etc. Also $\underline{a}\cdot(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\cdot\underline{c}$

so $\underline{\text{LHS} \Rightarrow 2(\underline{a}\times\underline{b})\cdot\underline{c} = \text{RHS}}$ ✓ 4

with $\underline{a} = (1,0,0)$ $\underline{b} = (1,1,0)$ $\underline{c} = (1,1,1)$

LHS $= ((2,1,0)\times(2,2,1))\cdot(2,1,1) = (1,-2,2)\cdot(2,1,1) = \underline{\underline{2}}$

RHS $= 2\,(0,0,1)\cdot(1,1,1) = \underline{\underline{2}}$ ✓ 2

(iii) $\underline{a}\times\underline{x} = \underline{b}$ cross with $\underline{a}$ $\Rightarrow$ $\underline{a}\times(\underline{a}\times\underline{x}) = \underline{a}\times\underline{b}$

$\Rightarrow (\underline{a}\cdot\underline{x})\underline{a} - a^2\underline{x} = \underline{a}\times\underline{b}$ 3

i.e. $3(1,2,1) - 6\underline{x} = (-9,12,-15)$

$6\underline{x} = (3,6,3)+(9,-12,15) \Rightarrow \underline{\underline{x} = 2,-1,3}$ 2

[Alt: $\underline{a}\times\underline{x} = \underline{b} \Rightarrow 2z-y=7,\ x-z=-1,\ y-2x=-5$
(& note last equation, say, is redundant) Also
$\underline{a}\cdot\underline{x} = 3 \Rightarrow x+2y+z=3 \Rightarrow$
$x=z-1,\ y=2z-7 \Rightarrow z=3,\ y=-1,\ x=2$ acceptable]

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Setter : F LEPPINGTON Setter's signature : F. Leppington
Checker : Checker's signature : [signature]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E5

PAPER 2

QUESTION

SOLUTION 6

Please write on this side only, legibly and neatly, between the margins

(i) The vector $v = (1,-2,3)$ is in P, and v is orthogonal to P. Hence the distance from P to the origin is $\sqrt{1+4+9} = \sqrt{14}$.

4 marks 4

(ii) The vector $u = (1,-\alpha,0)$ is orthogonal to Q. The planes P and Q are orthogonal if and only if v and u are orthogonal. We have $v.u = 1+2\alpha = 0$, thus $\alpha = -\frac{1}{2}$.

5 marks 5

(iii) To find a common point of P and Q set $x=1$. Then $y=-2$ and $z=3$, so that $v=(1,-2,3)$ belongs to both P and Q. We can take $b=v$. Now the vector a is any non-zero solution of $2x+y = x-2y+3z = 0$. Setting $x=1$ we find $y=-2$, $z=-\frac{5}{3}$. Thus we can take $a = (1,-2,-\frac{5}{3})$.

6 marks 6

Setter : Skorobogatov Setter's signature :

Checker : WILSON Checker's signature : J. Wilson

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E6

PAPER 2

QUESTION

SOLUTION 7

Please write on this side only, legibly and neatly, between the margins

We have $A^2 = \begin{pmatrix} -1 & a \\ -a & a^2-1 \end{pmatrix}$, $A^3 = \begin{pmatrix} -a & a^2-1 \\ 1-a^2 & a^3-2a \end{pmatrix}$,

$$A^4 = \begin{pmatrix} 1-a^2 & a^3-2a \\ -a^3+2a & a^4-3a^2+1 \end{pmatrix}.$$

3 marks Thus $A^3 = I$ if and only if $a = -1$. 3

3 marks Similarly, $A^4 = I$ if and only if $a = 0$. 3

2 marks For $a = -1$ we have $A^{-1} = A^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. 2

2 marks For $a = 0$ we have $A^{-1} = A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. 2

2 mark A^{-1} exist for all a since $\det A = 1$. 2

3 marks $A = A^{-1}$ implies that $A^2 = I$. This is not possible as $-1 \neq 1$. Contradiction. 3

Setter : Skorobogatov Setter's signature :

Checker : WILSON Checker's signature :

(i)(a)
$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & 3 & \lambda & 7 \\ 1 & 1 & 2 & \mu \end{array}\right) \begin{array}{c} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \to \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 5 & \lambda-6 & 5 \\ 0 & 2 & -1 & \mu-1 \end{array}\right)$$

$$5R_3-2R_2 \to \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 5 & \lambda-6 & 5 \\ 0 & 0 & 7-2\lambda & 5\mu-15 \end{array}\right)$$

$\therefore$ Infinitely many solutions when $\lambda = \frac{7}{2}$ and $\mu = 3$. 5

Let $z = k$, then $5y - \frac{5}{2}k = 5$, $x - y + 3k = 1$

$\therefore y = 1 + \frac{1}{2}k$, $x = 1 + 1 + \frac{1}{2}k - 3k = 2 - \frac{5}{2}k$

$$\therefore (x,y,z) = (2,1,0) + \frac{k}{2}(-5,1,2)$$ 4

(b): $\lambda = \frac{7}{2} + 5\alpha$, $\mu = 3 - 2\beta$ $\quad \therefore \left(\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 0 & 5 & 5\alpha-\frac{5}{2} & 5 \\ 0 & 0 & -10\alpha & -10\beta \end{array}\right)$ 2

$\therefore z = \beta/\alpha$, $y = 1 - (\alpha - \frac{1}{2})\beta/\alpha$

$x = 1 + y - 3z = 2 - (\alpha - \frac{1}{2})\beta/\alpha - 3\beta/\alpha = 2 - (\alpha + \frac{5}{2})\beta/\alpha$

$$\therefore (x,y,z) = (2,1,0) + \frac{\beta}{\alpha}\left(-(\alpha+\tfrac{5}{2}), -(\alpha-\tfrac{1}{2}), 1\right)$$ 4

15

Setter : HERBERT
Checker : S. REICH
Setter's signature : Dr C Herbert.
Checker's signature : S Reich

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E8 (i)

PAPER I. 2

QUESTION

SOLUTION 9(i)

Please write on this side only, legibly and neatly, between the margins

9(i) SOLUTION

$$\frac{dy}{dx} + \left(\frac{1}{x}+1\right)y = 1$$

Integrating factor $I = \exp\left\{\int\left(\frac{1}{x}+1\right)dx\right\} = e^{\ln x + x} = xe^{x}$

O.D.E $\Rightarrow \frac{d}{dx}\left(xe^{x}y\right) = xe^{x}$ 3

Soln: $xe^{x}y = \int \underset{u}{x}\,\underset{v'}{e^{x}}\,dx = xe^{x} - \int e^{x}dx + K = (x-1)e^{x} + K$ 3

$y = 1$ at $x = 2 \Rightarrow K = e^{2}$

So $\underline{xe^{x}y = (x-1)e^{x} + e^{2}}$ 1

Setter : F. LEPPINGTON Setter's signature : F.G. Leppington

Checker : WILSON Checker's signature : J. Wilson

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2001-2002

E8 (ii)

PAPER 1.2

QUESTION

Please write on this side only, legibly and neatly, between the margins

SOLUTION 9(ii)

$$x\frac{dy}{dx} = y + \frac{y^2}{1+x^2}$$

Introduce $v = y/x \Rightarrow v' = \frac{y'}{x} - \frac{y}{x^2}$ 1

$$\therefore \quad \frac{dv}{dx} = \frac{y^2/x^2}{1+x^2} = \frac{v^2}{1+x^2} .$$ 2

Hence, $\frac{dv}{v^2} = \frac{dx}{1+x^2}$

$$\Rightarrow \quad -\frac{1}{v} = \mathrm{Tan}^{-1}x + \text{const}$$

or $y = \dfrac{x}{c - \mathrm{Tan}^{-1}x}$ 3

$y(1) = \frac{4}{\pi} \Rightarrow$

$$\frac{4}{\pi} = \frac{1}{c - \pi/4} \Rightarrow c = \pi/2$$

$$\therefore \quad y(x) = \frac{x}{\frac{\pi}{2} - \mathrm{Tan}^{-1}x} .$$ 2

Setter : J Elgin

Setter's signature : J Elgin

Checker : F LEPPINGTON

Checker's signature : F.G. Leppington

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 8 + e^{-x}$$

To find the CF consider 2

$$y'' + 4y' + 4y = 0$$

try $y = Ae^{mx}$. $m^2 + 4m + 4 = 0$ 1

$(m+2)(m+2) = 0$

$\therefore\ y = (Ax+B)e^{-2x}$ 2

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 8$ Clearly PI is $y = 2$ 1

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x}$ Try $\underline{y = Ae^{-x}}$ $A - 4A + 4A = 1$ 1

$\therefore\ \underline{A = 1}$

$\therefore$ G.S. is $y = (Ax+B)e^{-2x} + 2 + e^{-x}$ 1

$\frac{d^2y}{dx^2} + 4y = \cos 2x$. CF Satisfies $\frac{d^2y}{dx^2} + 4y = 0$

$\therefore\ y = A\cos 2x + B\sin 2x$ 2

Since $\cos 2x$ is a CF try as the P.I. 2

$y = Cx\sin 2x + Dx\cos 2x$

$y' = C\sin 2x + 2Cx\cos 2x + D\cos 2x - 2Dx\sin 2x$

$y'' = 2C\cos 2x + 2C\cos 2x - 4Cx\sin 2x - 2D\sin 2x - 2D\sin 2x - 4Dx\cos 2x$ 2

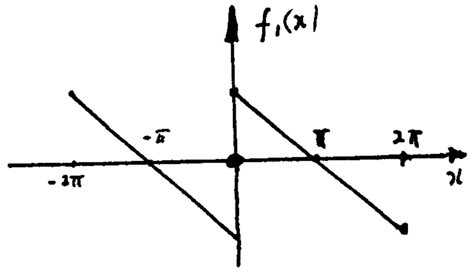
$\therefore\ 4C\cos 2x - \cancel{4Cx\sin 2x} - 4D\sin 2x - \cancel{4Dx\cos 2x}$

$+ \cancel{4Dx\cos 2x} + \cancel{4Cx\sin 2x} = \cos 2x$ $\therefore\ D = 0$ $C = 1/4$ 1

So PI is $\frac{1}{4}x\sin 2x$. General soln is $\therefore\ y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$ 1

Setter : J. R. CASH Setter's signature : JRCash

Checker : C J RIDLER-Rowe Checker's signature : CJRidler-Rowe



Given in data sheet:

$$f_1 = \frac{a_0}{2} + \sum_1^\infty a_n \cos\frac{n\pi x}{L} + \sum b_n \sin\frac{n\pi x}{L}$$

where $a_n = \frac{1}{L}\int_{-L}^{L}\cdots$, $b_n = \frac{1}{L}\int_{-L}^{L}\cdots dx$ — Graph 2

With $L=\pi$ here & $f_1 =$ odd, $\underline{a_n = 0}$ & $b_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx\, dx$. — 1

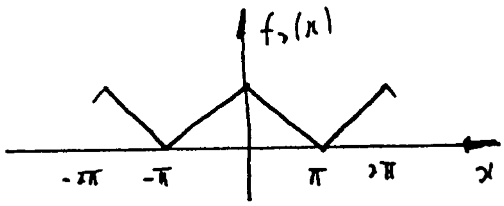
i.e. $b_n = \frac{2}{\pi}\left\{-\left[(\pi-x)\frac{\cos nx}{n}\right] - \frac{1}{n}\int_0^\pi \frac{\cos nx}{n}\right\} = \frac{2}{\pi}\left[-(\pi-x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^\pi$

$\Rightarrow b_n = \frac{2}{\pi}\cdot\frac{\pi}{n} = \frac{2}{n}$ & $\underline{f_1 = \sum_1^\infty \frac{2}{n}\sin nx}$ — 3

At $x=0$ series converges to $\underline{\underline{0}}$; At $x = \frac{\pi}{2}$ series cgs to $f_1(\pi/2) = \underline{\underline{\pi/2}}$ — 1

Thus $\frac{\pi}{2} = \sum_1^\infty \frac{2}{n}\sin\frac{n\pi}{2} = \sum_0^\infty \frac{2}{2m+1}\sin(m+\tfrac{1}{2})\pi = \sum_0^\infty \frac{2}{(2m+1)}(-1)^m$

$\Rightarrow \underline{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots}$ — 2



With f_2 even, $b_n \equiv 0$ — Graph 1

$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx\, dx$

$n=0$: $a_0 = \frac{2}{\pi}\int_0^\pi (\pi - x)\,dx = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \underline{\pi}$ — 1

$n \neq 0$: $a_n = \frac{2}{\pi}\left[(\pi-x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^\pi = \frac{2}{\pi}\,\frac{1-\cos n\pi}{n^2}$ — 3

$= 0$ if n even (but $\neq 0$), & $= \frac{4}{\pi n^2}$ if n is odd

so $f_2(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{\text{odd } n}\frac{\cos nx}{n^2}$ — 1

Alt form of sum $\frac{4}{\pi}\sum_{m=0}^\infty \frac{\cos(2m+1)x}{(2m+1)^2}$

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Setter : F G LEPPINGTON Setter's signature : F G Leppington
Checker : Checker's signature : [signature]