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UNIVERSITY OF LONDON

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B.ENG. AND M.ENG. EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1

Wednesday 4th June 2003 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Define what it means to say that a function f is odd or even, and give an example of each.

   (ii) Classify the following functions as odd, even or neither:

   (a) e^{-x} ;

   (b) $x \sin x$;

   (c) $x^2 \sin x$;

   (d) $2x/(x^2 - 1)$.

   (iii) Let $f(x) = e^x$ and $g(x) = 1/x^2$. Find $f(g(x))$ and $g(f(x))$. Find also the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$.

   (iv) Write

$$f(x) = \frac{2x}{x+1}$$

   as the sum of an even function and an odd function.

2. Let

$$f(x) = \frac{x(x+1)}{x-2} .$$

Find the stationary points of $f(x)$. By examining the sign of $f'(x)$ or otherwise, find which of these are maxima or minima.

Sketch the graph of $f(x)$, indicating clearly any asymptotes.

PLEASE TURN OVER

3. Find $\dfrac{dy}{dx}$ in each of the following cases. (In case (iv) you may express your answer in terms of x and y.)

(i)
$$y = \frac{x\,e^x}{\ln x}\ ;$$

(ii)
$$y = \ln\left(x + (x^2+1)^{1/2}\right)\ ;$$

(iii)
$$y = x^{\ln x}\ ;$$

(iv)
$$x + y + e^{xy} = 1\ .$$

4. (i) Show that
$$\frac{d}{dx}\sin x = \sin\left(x + \frac{\pi}{2}\right)$$
and generally, for $n \geq 1$,
$$\frac{d^n}{dx^n}\sin x = \sin\left(x + n\frac{\pi}{2}\right)\ .$$

(ii) Consider $y(x) = e^{x^2/2}$. Show that $\dfrac{dy}{dx} = xy$.

By differentiating this equation n times using Leibniz's formula, show that
$$y^{(n+1)}(x) = xy^{(n)}(x) + ny^{(n-1)}(x)\ .$$

Hence, or otherwise, evaluate $y^{(5)}(0)$.

(iii) The period T of small oscillations of a pendulum of length x is given by
$$T = 2\pi\sqrt{\frac{x}{g}}\ .$$

By using the formula
$$\frac{dT}{dx} = \lim_{\delta x\to 0}\frac{T(x+\delta x) - T(x)}{\delta x}\ ,$$
show that if there is a small manufacturing error δx in the length x, producing an error of 1% (so that $\delta x/x = 1/100$), then the error in T is approximately 0.5%.

5. Evaluate the following limits:

(i) $$\lim_{x\to 1} \frac{(x-2)(x+2)}{(x-3)(x+1)} \ ;$$

(ii) $$\lim_{x\to 0} \ \frac{1-\cos x}{\tan^2 x} \ ;$$

(iii) $$\lim_{x\to 0} \ x^x \ ;$$

(iv) $$\lim_{x\to -2} \ \frac{\sqrt{-2x}-2}{x+2} \ .$$

6. Evaluate the following integrals :

(i) $$\int \frac{\sinh^{-1}(x)}{(1+x^2)^{1/2}}\,dx \ ;$$

(ii) $$\int_0^{1/4} \ (\sinh x \cosh x)^2\,dx \ ;$$

(iii) $$\int \frac{dx}{1-\cos x} \ .$$

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7. (i) Express the function

$$\frac{x+1}{x^2 - x - 12}$$

in partial fraction form, and hence find

$$\int \frac{x+1}{x^2 - x - 12}\, dx\,.$$

(ii) Given that

$$I_n \;=\; \int_0^{\pi} e^x \sin^n x\, dx, \qquad (n = 0,\, 1,\, \ldots)\,,$$

show that

$$(n^2+1)\, I_n \;=\; n(n-1)\, I_{n-2}\,.$$

Hence verify that

$$I_5 \;=\; \frac{3}{13}\,(e^{\pi}+1)\,.$$

8. (i) Find the first four derivatives of the function $\ln(1+x)$.

Show that $\ln(1+x)$ has Maclaurin expansion

$$x \;-\; \frac{x^2}{2} \;+\; \frac{x^3}{3} \;+\; R_4$$

and find the form of the remainder R_4 for this function.

Use the first three terms of the above expansion to find an approximate value for

$$\int_{x=0}^{1} \frac{\ln(1+x)}{x}\, dx$$

and use the remainder term R_4 to give a bound for the error.

(ii) Find the radius of convergence of each of the following power series:

$$\text{(a)} \quad \sum_{n=0}^{\infty} n\, x^n\,; \qquad \text{(b)} \quad \sum_{n=0}^{\infty} \frac{n^2}{2^n}(x-1)^n\,.$$

9. (i) Express each of the following in the form $a + ib$:

(a) $(3+2i)\,(1-4i)$; (b) $\dfrac{7+6i}{1+3i}$; (c) $\left(\dfrac{1+\sqrt{3}i}{2}\right)^{104}$.

(ii) Describe and sketch the regions in the complex plane where

(a) $|z^2| \;=\; 5|z|$; (b) $|z-i| > |z+i|$.

(iii) Using de Moivre's theorem (or otherwise), find an expression for $\cos 4\theta$ as a polynomial in powers of $\cos\theta$.

10. (i) (a) Define the functions $\sin z$, $\cos z$ (where z is a complex number) in terms of the exponential function.

(b) Find all complex roots of the equation

$$\tan z \;=\; 2i\,.$$

(ii) (a) If $z \;=\; x + iy$, find the real and imaginary parts of $\sin(z^2)$ in terms of trigonometric and hyperbolic functions involving x and y.

(b) Hence find all complex numbers such that $\sin(z^2)$ is real.

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b\,;$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b\,.$$

$$\cos iz = \cosh z\,;\quad \cosh iz = \cos z\,;\quad \sin iz = i\sinh z\,;\quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \binom{n}{1} Df\,D^{n-1}g + \ldots + \binom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a,b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$. Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2 \ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$