EXAMINATION QUESTIONS/SOLUTIONS	SESSION 2002-2003	COURSE I (1)
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	Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please	QUESTION NO.

SOLUTION NO. 1

MARKSCHEME

(i) f is even if $f(x) = f(-x)$ for all x;

f is odd if $f(x) = -f(-x)$ for all x.

Examples: $f(x) = x^2$ is even ; $f(x) = x$ is odd — **2**

(ii) e^{-x} : neither

$x \sin x$: even

$x^2 \sin x$: odd

$2x/(x^2-1)$: odd. — **4**

(iii) $f(g(x)) = e^{1/x^2}$, $g(f(x)) = e^{-2x}$. — **2**

$f^{-1}(x) = \ln x$, $g^{-1}(x) = x^{-\frac{1}{2}}$. — **2**

(iv) In general, we can write

$$f(x) = \underbrace{\tfrac{1}{2}\left(f(x) + f(-x)\right)}_{\text{even}} + \underbrace{\tfrac{1}{2}\left(f(x) - f(-x)\right)}_{\text{odd}}.$$

When $f(x) = \dfrac{2x}{x+1}$, this gives

$$\frac{2x}{x+1} = \frac{-2x^2}{1-x^2} + \frac{2x}{1-x^2}.$$

— **5**

PAPER

QUESTION

SOLUTION 2

$$f(x) = \frac{x(x+1)}{x-2} = \frac{x^2+x}{x-2}$$

To find behavior as $x \to \pm\infty$:

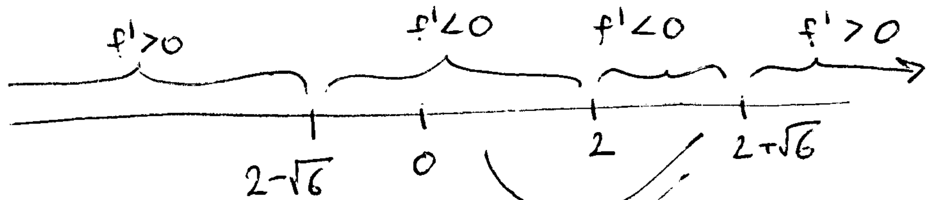
$$\frac{x^2+x}{x-2} = x+3+\frac{6}{x-2}$$ 2

Then: $f'(x) = \dfrac{(2x+1)(x-2)-(x^2+x)}{(x-2)^2} = \dfrac{x^2-4x-2}{(x-2)^2}$ 2

So $f'(x)=0 \iff x = 2\pm\sqrt{6}$ 2

$f(2+\sqrt{6}) = \dfrac{(2+\sqrt{6})(3+\sqrt{6})}{\sqrt{6}} = 5+\sqrt{24}$, $f(2-\sqrt{6}) = 5-\sqrt{24}$ 2

So



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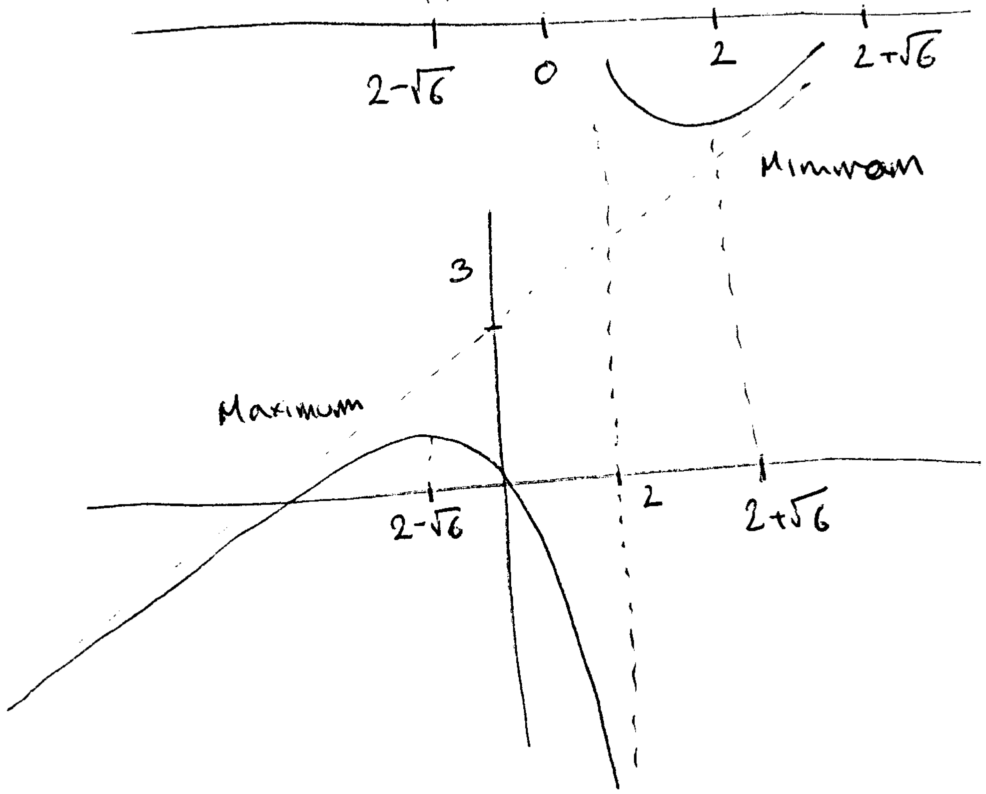

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EXAMINATION QUESTIONS/SOLUTIONS	SESSION 2002-2003	COURSE I (1)
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		QUESTION NO.
		SOLUTION NO. 3
		MARKSCHEME

(i) $$\frac{dy}{dx} = \frac{(xe^x)' - (xe^x)\frac{1}{x}}{(\ln x)^2} =$$

$$\frac{(e^x + xe^x)\ln x - e^x}{(\ln x)^2} = \frac{e^x(\ln x + x\ln x - 1)}{(\ln x)^2}.$$

4

(ii) $$\frac{dy}{dx} = \frac{1 + 2x(\frac{1}{2})(x^2+1)^{-1/2}}{x + (x^2+1)^{1/2}} =$$

$$(x^2+1)^{-1/2}\frac{(x^2+1)^{1/2} + x}{x + (x^2+1)^{1/2}} = (x^2+1)^{-1/2}.$$

3

(iii) $$\ln y = \ln x . \ln x = (\ln x)^2 .$$

$$\therefore \frac{dy}{dx}\frac{1}{y} = 2(\ln x)\frac{1}{x} .$$

$$\therefore \frac{dy}{dx} = 2\ln x . x^{\ln x - 1} .$$

4

(iv) $$1 + \frac{dy}{dx} + \left(\frac{d}{dx}(xy)\right)e^{xy} = 0 .$$

$$1 + \frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right)e^{xy} = 0 .$$

$$\therefore \frac{dy}{dx} = -\frac{ye^{xy} + 1}{xe^{xy} + 1} .$$

4

PAPER I.1

QUESTION

SOLUTION 4

(i) $\sin(x+\frac{\pi}{2}) = \sin x \cos\pi/2 + \cos x \sin\pi/2 = \cos x = \frac{d}{dx}\sin x.$ 2

If $\frac{d^n}{dx^n}\sin x = \sin(x+n\frac{\pi}{2})$ then $\frac{d^{n+1}}{dx^{n+1}}\sin x = \cos(x+n\pi/2)$.

But $\sin(x+(n+1)\frac{\pi}{2}) = \sin((x+n\frac{\pi}{2})+\frac{\pi}{2}) = \cos(x+n\frac{\pi}{2})$.

so result holds for $n+1$. Hence by induction the result follows for $n \geq 1$. [OR Use careful "and so on..." argument.] 3

(ii) $y' = (2x/2)e^{x^2/2} = xy.$ 1

Applying Leibniz's formula

$$y^{(n+1)} = xy^{(n)} + {}^nC_1 . 1 . y^{(n-1)} + 0 = xy^{(n)} + ny^{(n-1)}.$$ 3

Putting $x=0$ gives $y^{(n+1)} = ny^{(n-1)}$ so that

$y^{(5)}(0) = 4y^{(3)}(0)$ from taking $n=4$

$= 4.2\,y'(0)$

$= 0$ since $y'(0)=0$. 2

[OR note that $y(x)$ is even so that $y^{(5)}(0)=0$, as an odd order derivative.]

(iii) $\delta T = T(x+\delta x) - T(x)$

$\simeq \frac{dT}{dx}\delta x$

$= \frac{\pi}{\sqrt{xg}}\delta x.$

$\therefore \frac{\delta T}{T} \simeq \frac{\pi}{\sqrt{xg}}\frac{1}{2\pi}\sqrt{\frac{g}{x}}\,\delta x = \frac{1}{2}\frac{\delta x}{x} = \frac{1}{200}.$ 3

Hence the error in T is $\simeq 0.5\%$. 1

alteration

$\frac{\pi}{\sqrt{xg}}$

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Checker : CASH Checker's signature : [signature]

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SOLUTION NO. 5

(i) $\lim_{x \to 1} \frac{(x-2)(x+2)}{(x-3)(x+1)} = \frac{(-1)(3)}{(-2)(2)} = \frac{3}{4}$ — MARKSCHEME 2

(ii) $\lim_{x \to 0} \frac{1-\cos x}{\tan^2 x} = \lim_{x \to 0} \frac{\sin x}{2\tan x \sec^2 x}$

$= \lim_{x \to 0} \frac{\cos x}{2\sec^4 x + 2\tan x \, \frac{d}{dx}(\sec^2 x)}$ — 5

$= \frac{1}{2+0} = \frac{1}{2}.$

(iii) Let $y = x^x$, $\ln y = x \ln x$.

As $x \to 0$, $\ln y \to 0$, hence $y \to 1$, — 4

i.e. $\lim_{x \to 0} x^x = 1$.

(iv) $\lim_{x \to -2} \frac{\sqrt{-2x} - 2}{x+2} = \lim_{x \to -2} \frac{(\sqrt{-2x}-2)(\sqrt{-2x}+2)}{(x+2)(\sqrt{-2x}+2)}$ — 4

$= \lim_{x \to -2} \frac{-2x-4}{(x+2)(\sqrt{-2x}+2)} = \lim_{x \to -2} \frac{-2}{\sqrt{-2x}+2} = -\frac{1}{2}$

Solution.

(i) Use substitution $u = \sinh^{-1} x$ and

$$du = \frac{1}{(1+x^2)^{1/2}} dx$$

to obtain

$$\int \frac{\sinh^{-1} x}{(1+x^2)^{1/2}} dx = \int u\, du = \frac{1}{2} u^2 + C = \frac{1}{2} (\sinh^{-1} x)^2 + C.$$ 4

(ii) Using standard trigonometric identities:

$$\begin{aligned} \int (\sinh x \cosh x)^2 \, dx &= \int \left(\frac{1}{2}\sinh 2x\right)^2 dx \\ &= \frac{1}{4} \int \frac{1}{2} (\cosh 4x - 1)\, dx \\ &= \frac{1}{8} \left(\frac{1}{4}\sinh 4x - x\right) + C. \end{aligned}$$ 6

Hence

$$\int_0^{1/4} (\sinh x \cosh x)^2 \, dx = \frac{1}{8} \left[\frac{1}{4}\sinh 4x - x\right]_{x=0}^{x=1/4} = \frac{1}{32} (\sinh 1 - 1)$$

(iii) Use substitution $t = \tan(x/2)$ resulting in (formulae sheet):

$$\cos x = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}.$$

Hence

$$\begin{aligned} \int \frac{dx}{1-\cos x} &= \int \frac{2dt}{1+t^2-(1-t^2)} \\ &= \int \frac{dt}{t^2} = -t^{-1} + C \\ &= -\frac{1}{\tan(x/2)} + C. \end{aligned}$$ 5

Setter : S. REICH Setter's signature : S Reich
Checker : HERBERT Checker's signature : Dr C Herbert

PAPER 1

QUESTION

SOLUTION 7

Solution.

(i) Put

$$\frac{x+1}{x^2-x-12} = \frac{x+1}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}.$$

Clearing the fraction gives $A = 5/7$ for $x = 4$ and $B = 2/7$ for $x = -3$. Hence 5

$$\int \frac{x+1}{x^2-x-12}dx = \frac{5}{7}\ln|x-4| + \frac{2}{7}\ln|x+3| + C.$$

(ii)

$$\begin{aligned} I_n &= \int_0^\pi e^x \sin^n x\, dx \\ &= \left[e^x \sin^n x\right]_0^\pi - n\int_0^\pi e^x \sin^{n-1} x \cos x\, dx \\ &= -n\left[e^x \sin^{n-1} x \cos x\right]_0^\pi + n\int_0^\pi e^x \left((n-1)\sin^{n-2} x\cos^2 x - \sin^n x\right) dx \\ &= n(n-1)I_{n-2} - n(n-1)I_n - nI_n = n(n-1)I_{n-2} - n^2 I_n. \end{aligned}$$

4

Putting $n = 5$ and $n = 3$ successively, we get

$$I_5 = \frac{20}{26}I_3 = \frac{20}{26}\frac{6}{10}I_1 = \frac{6}{13}I_1.$$

2

$$\begin{aligned} I_1 &= \int_0^\pi e^x \sin x\, dx \\ &= -\left[e^x \cos x\right]_0^\pi + \int_0^\pi e^x \cos x\, dx \\ &= e^\pi + 1 + \left[e^x \sin x\right]_0^\pi - I_1 \\ 2I_1 &= e^\pi + 1 \end{aligned}$$

3

giving

$$I_5 = \frac{3}{13}\left(e^\pi + 1\right).$$

1

15

Setter : S. REICH

Setter's signature : S. Reich

Checker : HERBERT

Checker's signature : D R Herbert

(i) $f(x) = \ln(1+x)$. $\quad f' = \frac{1}{1+x}$, $\quad f'' = -\frac{1}{(1+x)^2}$, $\quad f''' = \frac{2}{(1+x)^3}$,

$f^{(4)} = -\frac{2.3}{(1+x)^4}$. 2

$$\therefore \quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + R_4$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} + R_4$$

where $R_4 = \frac{f^{(4)}(\bar{x})}{4!}x^4$ for some $\bar{x}$ between 0 and x

$$= \frac{-6x^4}{(1+\bar{x})^4\, 4!}.$$ 2

Using the first 3 terms gives

$$\int_0^{1/2} \frac{\ln(1+x)}{x}\,dx \simeq \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3}\right)dx = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36}.$$ 2

with error $= \int_0^1 \frac{R_4}{x}\,dx = -\int_0^1 \frac{6x^3}{(1+\bar{x})^4\, 4!}\,dx$.

$$\therefore \quad |\text{error}| = \int_0^1 \frac{x^3}{4(1+\bar{x})^{\cancel{4}}}\,dx$$

$$< \int_0^1 \frac{x^3}{4}\,dx \quad \text{since } 1+\bar{x} > 1,$$

giving $|\text{error}| < 1/16$. 4

(ii) (a) Applying the Ratio Test,

$$\left|\frac{(n+1)^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}}\right| = \left|\frac{(n+1)\,x^{n+1}}{n\,x^n}\right| = \frac{n+1}{n}|x| \to |x| \quad \text{as } n \to \infty.$$

The series converges if the last limit is < 1 and diverges if it is > 1.

Hence the radius of convergence is 1. 2

(b) $$\left|\frac{(n+1)^{\text{th}}\text{ term}}{n^{\text{th}}\text{ term}}\right| = \left|\frac{(n+1)^2(x-1)^{n+1}}{2^{n+1}}\,\frac{2^n}{n^2(x-1)^n}\right| = \left(\frac{n+1}{n}\right)^2\frac{|x-1|}{2} \to \frac{|x-1|}{2}.$$

Hence by the Ratio Test the series converges if $\frac{|x-1|}{2} < 1$

i.e. $|x-1| < 2$, and diverges if $|x-1| > 2$.

Hence the radius of convergence is 2. 3

Setter : RIDLER-ROWE $\qquad$ Setter's signature : JSRidler-Rowe

Checker : CASH $\qquad$ Checker's signature : JR Cash

PAPER 1

QUESTION

SOLUTION 9

(i) (a) $(3+2i)(1-4i) = 3-10i-8i^2 = 11-10i$ 1

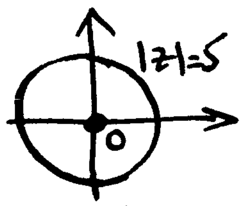
(b) $\dfrac{7+6i}{1+3i} = \dfrac{(7+6i)(1-3i)}{(1+3i)(1-3i)} = \dfrac{1}{10}(7-15i-18i^2)$

$= \dfrac{5}{2} - \dfrac{3}{2}i$ 2

(c) $\left(\dfrac{1+\sqrt{3}i}{2}\right)^{104} = \left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^{104} = \left(e^{i\pi/3}\right)^{104}$

$= e^{i\pi 34\frac{2}{3}} \equiv e^{i\pi\frac{2}{3}} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}.$ 2

(ii) $|z^2| = 5|z| \Rightarrow$ [$|z|=5$ CIRCLE CENTRE O RADIUS 5] AND [$|z|=0$. ORIGIN O.]



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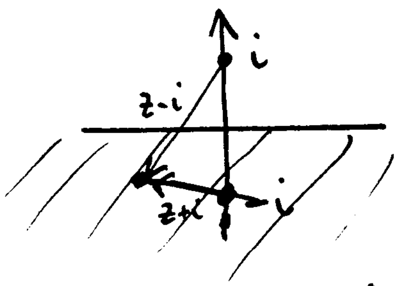
$|z-i| > |z+i|$ MODULUS $(z-i)$ > MODULUS $(z+i)$

$\Rightarrow \sqrt{x^2+(y-1)^2} > \sqrt{x^2+(y+1)^2}$

$\Rightarrow y<0.$

Lower $\frac{1}{2}$ plane



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(iii) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta.$

so $\cos 4\theta = \mathrm{Re}\left[(\cos\theta + i\sin\theta)^4\right]$

$= \mathrm{Re}\left[\cos^4\theta + 4i\sin\theta\cos^3\theta + 6i^2\sin^2\theta\cos^2\theta + 4i^3\sin^3\theta\cos\theta + i^4\sin^4\theta\right]$ 2

$= \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta$

$= \cos^4\theta - 6(1-\cos^2\theta)\cos^2\theta + (1-\cos^2\theta)^2$ 2

$= 1 - 8\cos^2\theta + 8\cos^4\theta.$

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Checker : J ELGIN Checker's signature : J Elgin

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2002 - 2003

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PAPER 1

QUESTION 1

SOLUTION 10

(i)(a) $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$, $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ 2

(b) $\tan z = \frac{\sin z}{\cos z} = \frac{\cancel{2}\left(e^{iz} - e^{-iz}\right)}{\cancel{2}i\left(e^{iz} + e^{-iz}\right)} = 2i$

$\therefore \frac{\left(e^{2iz} - 1\right)}{\left(e^{2iz} + 1\right)} = 2i^2 = -2$. and so $e^{2iz} = -1/3$

$\equiv \frac{1}{3} e^{i((2n+1)\pi)}$ 2

so $2iz = -\ln 3 + i(2n+1)\pi$. n an integer. 3

and $z = (2n+1)\frac{\pi}{2} + \frac{i}{2}\ln 3$.

(ii)(a) $z = x + iy \Rightarrow z^2 = (x^2 - y^2) + 2ixy$

$\sin z^2 = \sin(x^2-y^2)\cos(2ixy) + \cos(x^2-y^2)\sin(2ixy)$.

$\equiv \sin(x^2-y^2)\cosh(2xy) + i\cos(x^2-y^2)\sinh(2xy)$ 3

(b) For $\sin(z^2)$ to be real then $\cos(x^2-y^2)\sinh(2xy) = 0$.

So $\cos(x^2-y^2) = 0$ and/or $\sinh(2xy) = 0$.

$\downarrow$ $\downarrow$

$(x^2-y^2) = (2k+1)\frac{\pi}{2}$ (k integer). $x = 0$ and/or $y = 0$.

So z is $= \alpha$ (REAL) 2

or $= i\beta$ (PURE IMAGINARY)

or $= \pm\left[y^2 + \frac{(2k+1)}{2}\pi\right]^{1/2} + iy$. ($\alpha, \beta, y$ arbitrary). k integer. 3

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