E1.10 MATHS 1
(EE – 1st year)

UNIVERSITY OF LONDON

[I(1) 2005]

B.ENG. AND M.ENG. EXAMINATIONS 2005

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1

Wednesday 1st June 2005 10.00 am - 1.00 pm

Answer EIGHT questions.

Formulae sheet provided.

[Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.]

1. (i) Classify the following functions as odd, even or neither:

   (a) $(x^3 + 2x)\cos x$,

   (b) $(\cos x + \sin x)^2 - 1$,

   (c) xe^{x^3} .

   (ii) Compute $f(f(x))$ when $f(x) = \dfrac{5x+2}{3x-5}$.

   (iii) Find the inverse functions of $\sqrt{1-x}$ and $\ln\left(\dfrac{x+1}{x-1}\right)$.

2. Consider the function

$$f(x) = (x^2 - 4)\, e^{-x} .$$

   (i) Find the points where $f(x) = 0$.

   (ii) Find the vertical and horizontal asymptotes of f, if any.

   (iii) Use (i) and (ii) to determine the sign of $f(x)$, for all x.

   (iv) Find the points where $f'(x) = 0$.

   (v) Determine the local minima and maxima of f.

   (vi) Sketch the graph of f.

PLEASE TURN OVER

3. Find $\dfrac{dy}{dx}$ in terms of x in the following four cases, simplifying your answer where necessary :

(i)
$$y = \frac{\sin 2x}{x^2+2} \; ;$$

(ii)
$$y = \sinh^{-1}\left[\frac{3x}{4}\right] \; ;$$

(iii)
$$y = \ln\left[x + (1+x^2)^{1/2}\right] \; ;$$

(iv)
$$y = (\sin x)^x \; .$$

4. (i) If $x = t + \sin t$, $y = t + \cos t$, show that

$$(1+\cos t)^3 \, \frac{d^2y}{dx^2} = \sin t - \cos t - 1 \; .$$

(ii) Given that

$$x^2 + y^2 + xy = 1 \; ,$$

find dy/dx.

Given that $y = 1$ when $x = 0$, use the formula

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}$$

to find the approximate value of y at $x = 0.1$.

5. (i) You are given the limit

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 .$$

Use this to evaluate the following limits:

$$\lim_{\theta \to 0} \frac{\sin(\theta^2)}{(\sin\theta)^2} \qquad \text{and} \qquad \lim_{\theta \to \infty} \theta \sin(2/\theta) .$$

(ii) State l'Hôpital's Rule and use it to evaluate the limit:

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 + x - 6} .$$

6. (i) Use appropriate substitutions to evaluate the indefinite integrals

$$\int \frac{1}{x \ln x}\, dx \qquad \text{and} \qquad \int \sqrt{2 - x^2}\, dx .$$

(ii) Evaluate the integrals

$$\int_0^\infty (\cos nx)\, e^{-x}\, dx \qquad \text{and} \qquad \int_0^\infty (\sin nx)\, e^{-x}\, dx ,$$

in terms of n, where n is a given positive integer.

Hence show that

$$\lim_{n \to \infty} \int_0^\infty (\cos nx)\, e^{-x}\, dx = 0$$

and

$$\lim_{n \to \infty} \int_0^\infty (\sin nx)\, e^{-x}\, dx = 0 .$$

PLEASE TURN OVER

7. Let $I_n = \int_0^\infty x^{2n+1} e^{-x} \, dx$ where $n \geq 0$ is an integer.

(i) Find I_0 .

(ii) Show that $I_n = (2n+1)! \, I_0$.

(iii) Evaluate I_5 in terms of a factorial.

8. (i) Explain what it means for a series

$$\sum_{n=1}^{\infty} a_n$$

to be convergent.

(ii) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Hence, or otherwise, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \text{is also divergent .}$$

(iii) Find the radius of convergence R of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} .$$

9. (i) Using De Moivre's theorem, show that

$$\cos^6\theta = \frac{1}{32}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right] .$$

(ii) If

$$z_1 = -2 + 4i ,$$

$$z_2 = 2 + 2i ,$$

find $z_1 z_2$ and $\dfrac{z_1}{z_2}$ and show that $|z_1 z_2| = |z_1||z_2|$.

10. (i) Prove that

$$\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$$

and show that

$$|\cosh(x+iy)|^2 = \frac{\cosh 2x + \cos 2y}{2} .$$

(ii) If

$$y = \tanh^{-1} x ,$$

show that

$$\operatorname{sech} y = \sqrt{1-x^2}$$

and hence that

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) .$$

END OF PAPER

MATHEMATICS DEPARTMENT

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1, a_2, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3 TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2 f''(a)/2! + \ldots + h^n f^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!, \quad 0 < \theta < 1\,.$

(c) Taylor's expansion of $f(x, y)$ about (a, b):

$$f(a+h, b+k) = f(a, b) + [hf_x + kf_y]_{a,b} + 1/2!\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b} + \ldots$$

(d) Partial differentiation of $f(x, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u, v)$, $y = y(u, v)$, then $f = F(u, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

(e) Stationary points of $f(x, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let (a, b) be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a,b}$.
If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a maximum;
If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a minimum;
If $D < 0$ then (a, b) is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x, y)dx + Q(x, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2\ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x + 2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

a) i) odd; 1

ii) $(\cos x + \sin x)^2 - 1 = 2\cos x \sin x$, 2

so this function is also odd;

iii) neither. 2

b) $f(f(x)) = \dfrac{5(5x+2)+2(3x-5)}{3(5x+2)-5(3x-5)} = \dfrac{31x}{31} = x$. 4

c) Write $y = \sqrt{1+x}$, then $x = y^2 - 1$ is 2

the inverse function. If $y = \ln\left(\frac{x+1}{x-1}\right)$ then

$e^y = \dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1}$. It follows that

$x - 1 = \dfrac{2}{e^y - 1}$ which implies $x = \dfrac{e^y+1}{e^y-1}$. 4

Setter : A. Skorobogatov

Setter's signature : A Skorobogatov

Checker : LUZZATTO

Checker's signature : Stefano Luzzatto

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PAPER 1

QUESTION

SOLUTION 2

points:

(1) $f(x) = 0 \Leftrightarrow x^2 - 4 = 0$ or $\cancel{e^{-x} = 0}$

$\Leftrightarrow x = \pm 2$ — 2

(2) f is defined everywhere $\Rightarrow$ no vertical asymptotes — 1

$x \to +\infty \Rightarrow f(x) \to 0$, because the ~~exponential wins from the~~ polynomial / the exponential $\to 0$

$\Rightarrow$ horizontal asymptote $y = 0$ for $x \to +\infty$ — 1

$x \to -\infty \Rightarrow f(x) \to +\infty$, so no horizontal asymptote for $x \to -\infty$. — 1

(3) — 2

x		-2		2	
sign(f)	$+++$		$- - -$		$+++$

(4) $f'(x) = 2x e^{-x} - (x^2-4)e^{-x}$

$= e^{-x}(2x - x^2 + 4)$ — 3

$f'(x) = 0 \Leftrightarrow x^2 - 2x - 4 = 0$

$\Leftrightarrow x = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$ ($\approx -1, 3$)

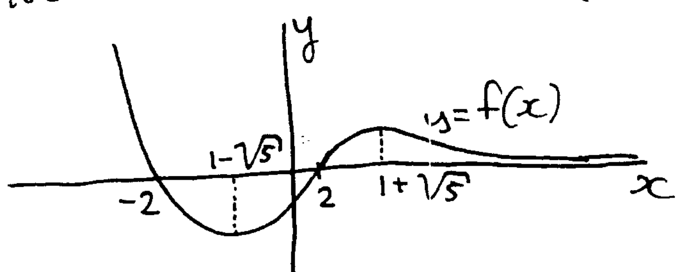
(5) — 2

x		$1-\sqrt{5}$		$1+\sqrt{5}$	
sign(f')	$- - -$	loc. min	$+ + + + + +$	loc. max	$- - -$

loc min $x = 1 - \sqrt{5}$: $f(x) = (6 - 2\sqrt{5})e^{-1+\sqrt{5}}$

loc max $x = 1 + \sqrt{5}$: $f(x) = (6 + 2\sqrt{5})e^{-1-\sqrt{5}}$

(6) — 3



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Setter : M. Noort — Setter's signature :

Checker : J.R. Casu — Checker's signature :

i) $y = \frac{\sin 2x}{x^2+2}$ $\quad \frac{dy}{dx} = \frac{2(x^2+2)\cos 2x - 2x\sin 2x}{(x^2+2)^2}$

by quotient rule // 2

ii) $y = \sinh^{-1}[3x/4]$

$\sinh y = 3x/4 \quad \cosh y \frac{dy}{dx} = 3/4$

$\frac{dy}{dx} = \frac{3}{4}\frac{1}{[1+\sinh^2 y]^{1/2}}$ 2

$= \frac{3}{4}\left[\frac{1}{\left[1+\frac{9x^2}{16}\right]^{1/2}}\right] = \frac{3}{[16+9x^2]^{1/2}}$ // 2 1

iii) $y = \ln[x+(1+x^2)^{1/2}]$. $\quad \frac{dy}{dx} = \frac{1+x(1+x^2)^{-1/2}}{x+(1+x^2)^{1/2}}$ 2

So $\frac{dy}{dx} = (1+x^2)^{-1/2}\left(\frac{(1+x^2)^{1/2}+x}{x+(1+x^2)^{1/2}}\right) = (1+x^2)^{-1/2}$ // 2

iv) $y = (\sin x)^x \implies \ln y = x\ln(\sin x)$ 1

$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x\frac{\cos x}{\sin x}$ 1

So $\frac{dy}{dx} = (\sin x)^x[\ln(\sin x) + x\cot x]$ // 2

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2004 -- 2005

Please write on this side only, legibly and neatly, between the margins

PAPER 1 (i)

QUESTION

SOLUTION 4

$x = t + \sin t, \quad y = t + \cos t \quad \therefore \frac{dx}{dt} = 1 + \cos t, \ \frac{dy}{dt} = 1 - \sin t$ 1, 1

$dy/dx = \frac{dy}{dt} \Big/ dx/dt = \frac{1 - \sin t}{1 + \cos t}$ 1

So $(1 + \cos t)\left(\frac{dy}{dx}\right) = 1 - \sin t$ 1

Differentiate again

$-\sin t \frac{dy}{dx} + (1 + \cos t)\frac{d^2y}{dx^2}\frac{dx}{dt} = -\cos t$ 2

$\therefore (1 + \cos t)^2 \, d^2y/dx^2 = \sin t\left(\frac{1 - \sin t}{1 + \cos t}\right) - \cos t$

$= \frac{\sin t - \sin^2 t - \cos t - \cos^2 t}{1 + \cos t}$ 2

$\therefore (1 + \cos t)^3 \, d^2y/dx^2 = \sin t - \cos t - 1$ 1

ii) $x^2 + y^2 + xy = 1$

$2x + 2y \, dy/dx + y + x \, dy/dx = 0$ 2

so $dy/dx = -\frac{(2x + y)}{x + 2y}$ 1

Formula is $y(x + \delta x) - y(x) \simeq \delta x \, y'(x)$ 1

So put $x = 0$, $\delta x = 0.1$.

$dy/dx\Big)_{x=0} = -(1)/2 = -\tfrac{1}{2}$

So $y(0.1) \simeq 1 - \tfrac{1}{2} \times 0.1 \simeq 1 - 0.05 \simeq 0.95$. 2

Setter : J. R. CASH Setter's signature : JR Cash

Checker : M v Noort Checker's signature : MvN

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Q5 (a) i) Using the fact that

$$\frac{\sin(\theta^2)}{(\sin\theta)^2} = \frac{\sin(\theta^2)}{\theta^2} \cdot \frac{\theta^2}{(\sin\theta)^2}$$ we find

$$\lim_{\theta\to 0} \frac{\sin(\theta^2)}{(\sin\theta)^2} = \lim_{\theta\to 0} \frac{\sin(\theta^2)}{\theta^2} \cdot \left(\lim_{\theta\to 0} \frac{\theta}{\sin\theta}\right)^2$$

$$= 1 \times 1^2 = 1,$$

using algebra of limits. [5]

ii) Since $\theta \cdot \sin(2/\theta) = \dfrac{\sin(2/\theta)}{2/\theta} \cdot 2$

we obtain

$$\lim_{\theta\to\infty} \theta \sin(2/\theta) = \lim_{\theta\to\infty} \frac{\sin(2/\theta)}{2/\theta} \cdot 2$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right) 2 = 2.$$ [4]

(b) L'Hôpital's rules states that if f and g ~~are differentiable functions~~ are such that $f(a) = g(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)},$$

provided $g'(a) \neq 0$. [3]

Setter : R. BEARDMORE Setter's signature : R. B

Checker : R CRASTER Checker's signature : RVC

Q5 (b) (cont'd)

Hence, in the case $f(x) = x^3 - 8$ and $g(x) = x^2 + x - 6$, we find $f(2) = 0$ and $g(2) = 0$. However,

$$f'(x) = 3x^2 \quad \text{and} \quad g'(x) = 2x + 1,$$

so that $g'(2) = 5 \neq 0$.

As a result,

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{3 \cdot 2^2}{5} = 12/5.$$

[3]

PAPER 1
QUESTION
SOLUTION 6a

Q6 (a) To find $\int \frac{1}{x \ln x} dx$, put $x = e^u$

so that $\int \frac{1}{x \ln x} dx = \int \frac{e^u du}{e^u . u} = \int \frac{du}{u}$

$$= \ln |u| + c,$$

$$= \ln |\ln x| + c \quad \text{for } x > 0.$$

[4]

To find $\int \sqrt{2 - x^2} dx$, put $x = \frac{\sqrt{2}}{1} \sin\theta$, then

$$\int \sqrt{2-x^2}\, dx = \int \sqrt{2 - 2\sin^2\theta}\, |\sqrt{2} \cos\theta| d\theta$$

$$= 2\int \cos^2\theta\, d\theta = \int 1 + \cos 2\theta\, d\theta$$

$$= \theta + \frac{\sin 2\theta}{2} = \theta + \sin\theta\cos\theta$$

$$= \sin^{-1}\left(x/\sqrt{2}\right) + \frac{x}{\sqrt{2}}\sqrt{1 - x^2/2}.$$

[6]

Setter : R. BEARDMORE
Setter's signature : R. Bdl
Checker : R. CRASTER
Checker's signature : RVC

Q6(b) If. $C_n := \int_0^\infty e^{-x} \cos nx \, dx$ and

$S_n := \int_0^\infty e^{-x} \sin nx \, dx$ then

$$C_n = \left[-e^{-x}\cos nx\right]_0^\infty - \int_0^\infty e^{-x} \sin nx \, dx \cdot n$$

$$= 1 - nS_n$$

and

$$S_n = \left[-e^{-x}\sin nx\right]_0^\infty + \int_0^\infty e^{-x} \cos nx \cdot dx \cdot n$$

$$= nC_n$$

Hence $C_n = 1 - n^2 C_n \Rightarrow C_n = \frac{1}{1+n^2}$

$\therefore \quad S_n = \frac{n}{1+n^2}$.

It is immediate that $\lim_{n\to\infty} C_n = \lim_{n\to\infty} S_n = 0$.

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PAPER 1
QUESTION
SOLUTION 7

i) $I_0 = \int_0^\infty x e^{-x}\,dx$

$$= [-xe^{-x}]_0^\infty + \int_0^\infty e^{-x}\,dx = 1$$ 5

ii) $I_n = [-x^{2n+1}e^{-x}]_0^\infty + (2n+1)\int_0^\infty x^{2n}e^{-x}\,dx$

$= (2n+1)[-x^{2n}e^{-x}]_0^\infty + (2n+1)(2n)\int_0^\infty x^{2n-1}e^{-x}\,dx$

$= (2n+1)(2n)\,I_{n-1}$ 5

Applying this recursively gives

$$I_n = (2n+1)!\,I_0$$ 3

iii) $I_5 = 11!$ in terms of factorial (SVH) 2

Setter : LUZZATTO Setter's signature : Stefano Luzzatto

Checker : A. Skorobogatov Checker's signature : [illegible]

(15)

PAPER 1

QUESTION

SOLUTION 8

i) A series is convergent if the limit of partial sums $S_n = a_1 + \cdots + a_n$ exists:

$$\lim_{n\to\infty} S_n = S < \infty.$$

4

ii) Use the integral test:

$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} \frac{1}{n}\,dn = [\ln n]_1^{\infty} = \infty$$

5

By the comparison test $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent since $\frac{1}{\sqrt{n}} > \frac{1}{n}$ for all $n \geq 2$.

2

iii) Applying the ratio test

$$\lim_{n\to\infty}\left| \frac{(-1)^{n+2} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1} x^n} \right| = |x|$$

4

So $R = 1$.

Setter: LUZZATTO   Setter's signature: Stefano Luzzatto

Checker: A. Skorobogatov   Checker's signature: [signature]

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PAPER

QUESTION

SOLUTION 9

(i)

Let $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta$

$2^6 \cos^6\theta = \left(z + \frac{1}{z}\right)^6$

$= z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right)$ 5

$+ 20 \quad (+)$

De Moivre.

$(\cos\theta + i\sin\theta)^n = z^n = \cos n\theta + i\sin n\theta$

$(\cos\theta - i\sin\theta)^n = \frac{1}{z^n} = \cos n\theta - i\sin n\theta$ 3

$\therefore (+) = 2\cos 6\theta + 2.6\cos 4\theta + 15.2.\cos 2\theta + 20$

$\therefore \cos^6\theta = \frac{1}{2^5}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$ 2

Setter : R CRASTER Setter's signature : RC

Checker : R. BEARDMORE Checker's signature : R. Bel

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(ii) If $z_1 = -2+4i \quad z_2 = 2+2i$

$z_1 z_2 = 2(-1+2i)(1+i)2 = 4(-3+i)$ 1

$\frac{z_1}{z_2} = \frac{2(-1+2i)(1-i)}{2(1+i)(1-i)} = \frac{1+3i}{2}$ 1

$|z_1| = 2\sqrt{1+2^2} = 2\sqrt{5}$

$|z_2| = 2\sqrt{1+1} = 2\sqrt{2}$ 2

$|z_1||z_2| = 4\sqrt{10}$

$|z_1 z_2| = 4\sqrt{1+3^2} = 4\sqrt{10}.$ 1

Setter : R CASTER Setter's signature : [signature]

Checker : R. BEARDMORE Checker's signature : R. [signature]

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(i) $\cosh(x+iy) = \cos(i(x+iy)) = \cos[ix-y]$

$= \cos ix \cos(+y) + \sin ix \sin y$

$= \cosh x \cos y \quad + i \sinh x \sin y$ 2

$|\cosh(x+iy)|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y$

$= \cosh^2 x \cos^2 y + (\cosh^2 x - 1)\sin^2 y$

$= \cosh^2 x - \sin^2 y$

$= \frac{1}{2}(\cancel{1}+\cosh 2x) - \frac{1}{2}[\cancel{1}-\cos 2y]$ 5

$= \cosh 2x + \cos 2y$

(ii) $\tanh y = x$

$\cosh^2 y - \sinh^2 y = 1$

$1 - \tanh^2 y = \operatorname{sech}^2 y$

$\sqrt{1-x^2} = \operatorname{sech} y = \frac{1}{\cosh y}$ 2

$\cosh y = \frac{1}{\sqrt{1-x^2}} \qquad \sinh y = \sqrt{1 + \frac{1}{1-x^2}}$ 2

$= \frac{\sqrt{1-1+x^2}}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

$e^y = \cosh y + \sinh y = \frac{x+1}{\sqrt{1-x^2}} = \sqrt{\frac{x+1}{1-x}}$ 3

$y = \frac{1}{2}\log\sqrt{\frac{x+1}{1-x}}$ 1