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UNIVERSITY OF LONDON [I(1) 2004]

B.ENG. AND M.ENG. EXAMINATIONS 2004

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City & Guilds of London Institute

PART I : MATHEMATICS 1

Wednesday 2nd June 2004 10.00 am - 1.00 pm

Answer EIGHT questions.

Corrected Copy

[*Before starting, please make sure that the paper is complete; there should be 7 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. (i) Define what it is meant by :

   (a) an even function,

   (b) an odd function.

   Let $f(x)$ be an odd function.

   What can you say about $f(x)^2$?

   (ii) Classify the following functions as odd, even or neither:

   (a) e^x ; (b) e^{x^2} ; (c) e^{x^3} ;

   (d) $\sin(x)$; (e) $\sin(x^2)$; (f) $\sin(x^3)$.

   (iii) Find the inverse function of

$$f(x) = \frac{x+1}{x+2} .$$

   (iv) Write $\dfrac{1}{x+2}$ as the sum of an even function and an odd function.

2. Let

$$f(x) = \frac{1}{x^2-1} .$$

   (i) Find the stationary point of $f(x)$.

   (ii) Find the second derivative of $f(x)$.

   Determine whether the stationary point is a maximum or a minimum.

   (iii) Sketch the graph of $f(x)$.

   Indicate clearly the vertical and horizontal asymptotes and the location of the stationary point.

PLEASE TURN OVER

3. Find $\dfrac{dy}{dx}$ in terms of x in the following three cases, simplifying your answer where necessary :

(i)
$$y = \frac{\sin x}{1 + \cos x} \ ;$$

(ii)
$$y = e^{x+x^2} \cosh x \ ;$$

(iii)
$$y = x^{\ln x} \ ;$$

Show that if $y = \tan^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

4. (i) Suppose that $x(t) = \cos^2 t$ and $y(t) = \tan t$. Find $\dfrac{dy}{dx}$ first in terms of t and then in terms of x and y.

Check your answer by finding a relation between x and y and then differentiating.

(ii) Suppose that $y - 2x + \sin(xy) = 0$.

Prove that $\dfrac{dy}{dx} = \dfrac{2 - y\cos(xy)}{1 + x\cos(xy)}$.

Given also that $y = 0$ when $x = 0$, use the formula

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x + \delta x) - y(x)}{\delta x}$$

to find the approximate value of y for $x = 0.01$.

5. (i) You are given that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 .$$

Evaluate the following limits:

(a)
$$\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta} ,$$

(b)
$$\lim_{\theta \to 0} \frac{(\sin(\theta))^2}{\theta} ,$$

(c)
$$\lim_{\theta \to 0} \frac{\sin(\theta^2)}{\theta \sin(\theta)} .$$

(ii) State whether or not the following limits exist; if so, give the limiting value:

(a)
$$\lim_{x \to 0} x \sin(1/x) ,$$

(b)
$$\lim_{x \to \infty} \left(\sqrt{1+x} - \sqrt{x}\right) ,$$

(c)
$$\lim_{x \to \infty} \sin(1/x) ,$$

(d)
$$\lim_{x \to 1} \frac{x-1}{x^n - 1} \quad \text{where } n \geq 2 \text{ is an integer} ,$$

(e)
$$\lim_{n \to \infty} \frac{2^n}{n!} \quad \text{where } n \text{ is an integer} .$$

PLEASE TURN OVER

6. (i) Using a trigonometric substitution, evaluate the indefinite integral

$$\int \frac{x}{\sqrt{1-x^2}}\, dx \ .$$

(ii) Given the definite integral

$$I_n \ = \ \int_0^{\pi/2} (\cos x)^n \, dx \ ,$$

where $n \geq 0$ is an integer, evaluate I_0 and I_1 and show that

$$I_n \ = \ \frac{n-1}{n}\, I_{n-2} \ , \qquad (n \geq 2) \ .$$

Hence evaluate

$$\int_0^{\pi/2} (\cos x)^3 \ (\sin x)^2 \ dx \ .$$

7. Evaluate the following indefinite integrals :

(i) $$\int \frac{4x-8}{x^2-4x+5}\, dx \ ;$$

(ii) $$\int \frac{\sec^2 x}{\tan x}\, dx \ ;$$

(iii) $$\int x^2 \ln x \, dx \ ;$$

(iv) $$\int \frac{x+1}{x^2-3x+2}\, dx \ .$$

8. (i) Decide whether each of the following series is convergent or divergent :

$$\text{(a)} \quad \sum_{1}^{\infty} \frac{2n+5}{100n} \, ; \qquad \text{(b)} \quad \sum_{1}^{\infty} \frac{n^{100}}{2^n+5} \, .$$

(ii) Find the radius of convergence of the following power series :

$$\text{(a)} \quad \sum_{1}^{\infty} \frac{|x-2|^n}{5^n} \, ; \qquad \text{(b)} \quad \sum_{1}^{\infty} \frac{(n+2)!}{(2n)!} \, x^n \, .$$

(iii) Find the first three non-zero terms of the Maclaurin series of $\ln(1+x)$.

9. (i) Write

(a) $5 + i12$ and (b) $1/(5 + i12)$

in polar form and indicate the position of each point in the complex plane, stating the tangent of the argument in each case.

(ii) De Moivre's theorem states

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

for any value of the real parameter n.

Use this to prove the trigonometric identities

$$\cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta = \cos(n+1)\theta \, ,$$

$$\cos(n\theta)\sin\theta + \sin n\theta\cos\theta = \sin(n+1)\theta \, .$$

PLEASE TURN OVER

10. (i) The hyperbolic functions $\cosh x$ and $\sinh x$ are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Use these to derive the hyperbolic form of de Moivre's theorem

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx .$$

(ii) Prove that

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y .$$

Hence or otherwise, find all the roots of the equation

$$\cosh^2 z = -1 ,$$

where $z = x + iy$.

END OF PAPER

M A T H E M A T I C S   D E P A R T M E N T

MATHEMATICAL FORMULAE

1. VECTOR ALGEBRA

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = (a_1,\, a_2,\, a_3)$$

Scalar (dot) product: $\mathbf{a}\,.\,\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Scalar triple product:

$$[\mathbf{a},\, \mathbf{b},\, \mathbf{c}] = \mathbf{a}\,.\,\mathbf{b}\times\mathbf{c} = \mathbf{b}\,.\,\mathbf{c}\times\mathbf{a} = \mathbf{c}\,.\,\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector triple product: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{a})\mathbf{c}$

2. SERIES

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots \quad (\alpha \text{ arbitrary, } |x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots\,,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \ldots\,,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n\frac{x^{n+1}}{(n+1)} + \ldots (-1 < x \le 1)$$

3. TRIGONOMETRIC IDENTITIES AND HYPERBOLIC FUNCTIONS

$$\sin(a+b) = \sin a\,\cos b + \cos a\,\sin b\,;$$

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\,.$$

$$\cos iz = \cosh z\,; \quad \cosh iz = \cos z\,; \quad \sin iz = i\sinh z\,; \quad \sinh iz = i\sin z\,.$$

4. DIFFERENTIAL CALCULUS

(a) Leibniz's formula:

$$D^n(fg) = f\,D^n g + \tbinom{n}{1} Df\,D^{n-1}g + \ldots + \tbinom{n}{r} D^r f\,D^{n-r}g + \ldots + D^n f\,g\,.$$

(b) Taylor's expansion of $f(x)$ about $x = a$:

$$f(a+h) = f(a) + hf'(a) + h^2f''(a)/2! + \ldots + h^nf^{(n)}(a)/n! + \epsilon_n(h)\,,$$

where $\epsilon_n(h) = h^{n+1}f^{(n+1)}(a+\theta h)/(n+1)!$, $\quad 0 < \theta < 1$.

(c) Taylor's expansion of $f(x,\, y)$ about $(a,\, b)$:

$$f(a+h,\, b+k) = f(a,\, b) + [hf_x + kf_y]_{a.b} + 1/2!\left[h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}\right]_{a.b} + \ldots$$

(d) Partial differentiation of $f(x,\, y)$:

i. If $y = y(x)$, then $f = F(x)$, and $\dfrac{dF}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$.

ii. If $x = x(t)$, $y = y(t)$, then $f = F(t)$, and $\dfrac{dF}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

iii. If $x = x(u,\, v)$, $y = y(u,\, v)$, then $f = F(u,\, v)$, and

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\,, \qquad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\,.$$

(e) Stationary points of $f(x,\, y)$ occur where $f_x = 0$, $f_y = 0$ simultaneously.
Let $(a,\, b)$ be a stationary point: examine $D = [f_{xx}f_{yy} - (f_{xy})^2]_{a.b}$.
If $D > 0$ and $f_{xx}(a,\, b) < 0$, then $(a,\, b)$ is a maximum;
If $D > 0$ and $f_{xx}(a,\, b) > 0$, then $(a,\, b)$ is a minimum;
If $D < 0$ then $(a,\, b)$ is a saddle-point.

(f) Differential equations:

i. The first order linear equation $dy/dx + P(x)y = Q(x)$ has an integrating factor $\quad I(x) = \exp[\int P(x)(dx]$, so that $\dfrac{d}{dx}(Iy) = IQ$.

ii. $P(x,\, y)dx + Q(x,\, y)dy = 0$ is exact if $\partial Q/\partial x = \partial P/\partial y$.

5. INTEGRAL CALCULUS

(a) An important substitution: $\tan(\theta/2) = t$:
$\sin\theta = 2t/(1+t^2)$, $\cos\theta = (1-t^2)/(1+t^2)$, $d\theta = 2\,dt/(1+t^2)$.

(b) Some indefinite integrals:

$$\int (a^2 - x^2)^{-1/2} dx = \sin^{-1}\left(\frac{x}{a}\right), \quad |x| < a.$$

$$\int (a^2 + x^2)^{-1/2} dx = \sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{\frac{x}{a} + \left(1 + \frac{x^2}{a^2}\right)^{1/2}\right\}.$$

$$\int (x^2 - a^2)^{-1/2} dx = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left|\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right|.$$

$$\int (a^2 + x^2)^{-1} dx = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right).$$

6. NUMERICAL METHODS

(a) Approximate solution of an algebraic equation:

If a root of $f(x) = 0$ occurs near $x = a$, take $x_0 = a$ and
$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$, $n = 0, 1, 2 \ldots$

(Newton Raphson method).

(b) Formulae for numerical integration: Write $x_n = x_0 + nh$, $y_n = y(x_n)$.

i. Trapezium rule (1-strip): $\int_{x_0}^{x_1} y(x)dx \approx (h/2)[y_0 + y_1]$.

ii. Simpson's rule (2-strip): $\int_{x_0}^{x_2} y(x)dx \approx (h/3)[y_0 + 4y_1 + y_2]$.

(c) Richardson's extrapolation method: Let $I = \int_a^b f(x)dx$ and let I_1, I_2 be two estimates of I obtained by using Simpson's rule with intervals h and $h/2$.

Then, provided h is small enough,

$$I_2 + (I_2 - I_1)/15,$$

is a better estimate of I.

7. LAPLACE TRANSFORMS

Function	Transform	Function	Transform
$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)dt$	$af(t) + bg(t)$	$aF(s) + bG(s)$
df/dt	$sF(s) - f(0)$	d^2f/dt^2	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$	$tf(t)$	$-dF(s)/ds$
$(\partial/\partial\alpha)f(t,\alpha)$	$(\partial/\partial\alpha)F(s,\alpha)$	$\int_0^t f(t)dt$	$F(s)/s$
$\int_0^t f(u)g(t-u)du$	$F(s)G(s)$		
1	$1/s$	$t^n (n = 1, 2\ldots)$	$n!/s^{n+1}$, $(s > 0)$
e^{at}	$1/(s-a)$, $(s > a)$	$\sin\omega t$	$\omega/(s^2+\omega^2)$, $(s > 0)$
$\cos\omega t$	$s/(s^2+\omega^2)$, $(s > 0)$	$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s, $(s, T > 0)$

8. FOURIER SERIES

If $f(x)$ is periodic of period $2L$, then $f(x+2L) = f(x)$, and

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3, \ldots$$

Parseval's theorem

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

PAPER	QUESTION	SOLUTION
I(1)200		1

(i) A function $f(x)$ is even if $f(x) = f(-x)$. 1

A function $f(x)$ is odd if $f(x) = -f(-x)$.

In that case $f(-x)^2 = (-f(x))^2 = f(x)^2$ 2

hence $f(x)^2$ is an even function.

(ii) (a) neither, (b) even, (c) neither,

(d) odd, (e) even, (f) odd. 6

(iii) Write $y = \frac{x+1}{x+2}$. Then $(y-1)x = 1-2y$

hence the inverse function is $\frac{1-2y}{y-1}$. 3

(iv) $\frac{1}{x+2} = \frac{1}{2}\left(\frac{1}{x+2} + \frac{1}{-x+2}\right) + \frac{1}{2}\left(\frac{1}{x+2} - \frac{1}{-x+2}\right) =$

$= -\frac{2}{x^2-4} + \frac{x}{x^2-4}$. Here 3

$-\frac{2}{x^2-4}$ is even, and $\frac{x}{x^2-4}$ is odd.

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Checker: S. LUTTATTO   Checker's signature: [signature]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

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PAPER 1

QUESTION

SOLUTION 2

a) $f'(x) = \dfrac{-2x}{(x^2-1)^2}$, for stationary 2

point: $f'(x) = 0$, hence $x = 0$ 2

b) $f''(x) = \dfrac{-2}{(x^2-1)^2} + \dfrac{8x^2}{(x^2-1)^3}$ 3

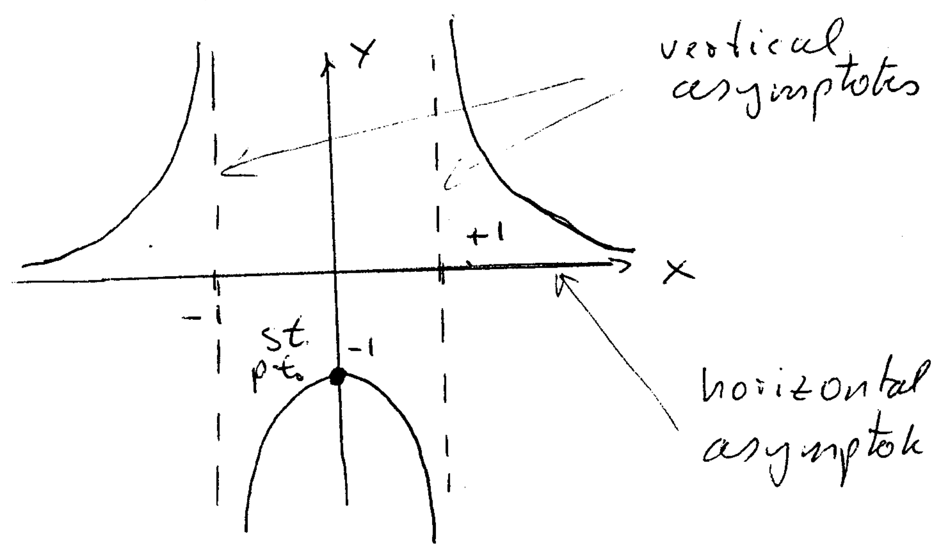
for $x = 0$, $f''(0) = -2 < 0$, stationary 2

point is maximum.

c) vertical asymptotes at $x = \pm 1$

horizontal asymptote is $y = 0$ 3

stationary point at $(0, -1)$



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(i) $$\frac{dy}{dx} = \frac{\cos x\,(1+\cos x) - \sin x\,(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x + \cos x}{(1+\cos x)^2}$$
$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}.$$ 4

(ii) $$\frac{dy}{dx} = (1+2x)\,e^{x+x^2}\cosh x + e^{x+x^2}\sinh x$$
$$= e^{x+x^2}\{(1+2x)\cosh x + \sinh x\}.$$ 3

(iii) $$\ln y = (\ln x)^2.$$
$$\frac{dy}{dx}\frac{1}{y} = \frac{2}{x}\ln x. \qquad \frac{dy}{dx} = \frac{2y}{x}\ln x = 2x^{-1+\ln x}\ln x.$$ 4

If $x = \tan y$. $\quad \frac{dx}{dy} = \sec^2 y = 1+\tan^2 y = 1+x^2.$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}.$$ 4

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Checker : J. R. CASH   Checker's signature : [signature]

(i) $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right) \Big/ \left(\dfrac{dx}{dt}\right) = \dfrac{\sec^2 t}{-2\cos t \sin t} = -\dfrac{1}{2\cos^3 t \sin t}$ 2

$\dfrac{dy}{dx} = -\dfrac{1}{2\cos^4 t \tan t} = -\dfrac{1}{2x^2 y}$. 2

$\tan^2 t = 1 + \sec^2 t$. $\therefore\ y^2 = 1 + \dfrac{1}{x}$, 2

$2y\dfrac{dy}{dx} = -\dfrac{1}{x^2}$. $\therefore\ \dfrac{dy}{dx} = -\dfrac{1}{2x^2 y}$. 2

(ii) $\dfrac{dy}{dx} - 2 + y\cos(xy) + x\dfrac{dy}{dx}\cos(xy) = 0$

giving $\dfrac{dy}{dx}$ as required. 3

$y(x+\delta x) \simeq y(x) + \left(\dfrac{dy}{dx}\right)\delta x$ $\therefore$ Putting $x = 0$, $\delta x = 0.01$,

$y(0.01) \simeq 0 + 2 \times 0.01 = 0.02$. 4

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A5 a) i) $\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta} = 2\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta} = 2.$ 5

ii) $\lim_{\theta\to 0}\frac{(\sin(\theta))^2}{\theta} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\lim_{\theta\to 0}(\sin\theta)^{1} = 1\cdot 0 = 0.$

iii) $\lim_{\theta\to 0}\frac{\sin(\theta^2)}{\theta\sin\theta} = \lim_{\theta\to 0}\frac{\sin(\theta^2)}{\theta^2}\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1.$

b) ~~i) $\lim_{x\to 0}\sin(1/x)$ does not exist as the function in this limit takes every value between -1 and 1 infinitely often in every interval of $x = 0$.~~

ii) Using $|x\sin(1/x)| \le |x|$ we obtain $\lim_{x\to 0} x\sin(1/x) = 0.$

iii) Using

$$\sqrt{1+x}-\sqrt{x} = (\sqrt{1+x}-\sqrt{x})\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} = \frac{1}{\sqrt{1+x}+\sqrt{x}}$$

we obtain $\lim_{x\to\infty}(\sqrt{1+x}-\sqrt{x}) = 0.$

iv) Using l'Hôpital's rule we find

$$\lim_{x\to 1}\frac{x-1}{x^n-1} = \lim_{x\to 1}\frac{1}{nx^{n-1}} = \frac{1}{n}.$$

$5\times 2 = 1$

v) $\lim_{x\to\infty}\sin(1/x) = \lim_{y\to 0}\sin(y) = 0.$

vi) Using

$$\frac{2^n}{n!} = \frac{2.2...2}{n(n-1)...1} = \frac{2}{n}.\frac{2}{n-1}...\frac{2}{2}\frac{2}{1} \le \frac{4}{n},$$

we have $\lim_{n\to\infty}\frac{2^n}{n!} = 0.$

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MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

PAPER 1

QUESTION

SOLUTION 6

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A6 a) To evaluate

$$I \equiv \int \frac{x}{\sqrt{1-x^2}}dx,$$

let $x = \sin\theta$ and then

$$I = \int \frac{\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta = \int \frac{\sin\theta\cos\theta}{|\cos\theta|}d\theta = \int \sin\theta d\theta = -\cos\theta,$$

provided $\cos\theta > 0$. Hence $I = -\sqrt{1-x^2}+c$. 5

b) If $I_n = \int_0^{\pi/2}(\cos x)^n dx$ then integrating by parts we find

$$I_n = \int_0^{\pi/2}(\cos x)^{n-1}\cos x dx = \sin x(\cos x)^{n-1}\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x(n-1)(\cos x)^{n-2}\cdot(-\sin x)dx.$$

so that

$$I_n = (n-1)\int_0^{\pi/2}(\sin x)^2(\cos x)^{n-2}dx = (n-1)\int_0^{\pi/2}(1-\cos^2 x)(\cos x)^{n-2}dx$$

and therefore

$$I_n = (n-1)(I_{n-2}-I_n) \Longrightarrow I_n(1+n-1) = (n-1)I_{n-2} \Longrightarrow I_n = \frac{n-1}{n}I_{n-2}.$$ 7

Now $I_0 = \int_0^{\pi/2} 1dx = \pi/2$ and $I_1 = \int_0^{\pi/2}\cos x dx = \sin(\pi/2) = 1$, and

$$\int_0^{\pi/2}(\cos x)^3(\sin x)^2 dx = \int_0^{\pi/2}(\cos x)^3(1-\cos^2 x)dx = I_3 - I_5.$$

But $I_3 = \frac{2}{3}I_1 = \frac{2}{3}$ and $I_5 = \frac{4}{5}I_3 = \frac{8}{15}$, so that

$$I_3 - I_5 = \frac{2}{3} - \frac{8}{15} = \frac{6}{45} = \frac{2}{15}.$$ 3

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Checker : J ELGIN

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PAPER I (1) 2004

QUESTION

SOLUTION 7

i) $$\int \frac{4x-8}{x^2-4x+5}\,dx = 2\int \frac{2x-4}{x^2-4x+5}\,dx$$

$$= 2\ln\left(x^2-4x+5\right) + C$$ 2

ii) $$\int \frac{\sec^2 x}{\tan x}\,dx = \ln \tan x + C$$ 2

iii) Let $u = \ln x$, $\frac{du}{dx} = 1/x$ 2

$v = x^3/3$, $\frac{dv}{dx} = x^2$

$$\int x^2 \ln x\,dx = \int u\,dv = uv - \int v\,du$$ 2

$$= \ln x\left(\frac{x^3}{3}\right) - \frac{1}{3}\int x^3 \, {}^1\!/_x\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + C$$

$$= \frac{x^3}{3}\left(\ln x - 1/3\right) + C$$ 2

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Setter's signature : Stefano Luzzatto

Checker: A. Skorobogatov

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PAPER I(1) 2004

QUESTION

SOLUTION 7 (cont)

(v)

$$\frac{x+1}{x^2-3x+2} = \frac{x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$ 2

$$x+1 = A(x-2) + B(x-1)$$ 2

$$\therefore \quad x = 1 \Rightarrow A = -2$$

$$x = 2 \Rightarrow B = 3$$

$$\int \frac{x+1}{x^2-3x+2}\,dx = \int \frac{3}{x-2}\,dx - \int \frac{2}{x-1}\,dx$$ 1

$$= 3\ln(x-2) - 2\ln(x-1) + C$$

Setter : LUZZATTO Setter's signature : [signature]

Checker : Skorobogatov Checker's signature : [signature] i)

(a) Terms do not tend to zero, so series diverges. 1

(b) Ratio test.

$$\frac{|U_{n+1}|}{|U_n|} = \frac{(n+1)^{100}}{2^{n+1}+5} \cdot \frac{2^n+5}{n^{100}} = \left(\frac{n+1}{n}\right)^{100} \cdot \frac{2^n+5}{2^{n+1}+5}$$ 2

$$= \left(\frac{n+1}{n}\right)^{100} \frac{1+5/2^n}{2+5/2^n}$$ 2

$$\to 1/2 < 1 \quad \text{converges}$$ 1

ii) (a) $$\frac{|U_{n+1}|}{|U_n|} = \frac{(x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(x-2)^n} = \frac{|x-2|}{5}$$ 1

$$< 1 \iff |x-2| < 5 \qquad R = 5$$ 2

(b) $$\frac{|U_{n+1}|}{|U_n|} = \frac{(n+3)!}{(2n+2)!} \cdot \frac{2n!}{(n+2)!} \cdot \frac{x^{n+1}}{x^n}$$ 1

$$= \frac{n+3}{(2n+1)(2n+2)} x = \frac{1}{2(2n+1)} x \to 0$$ 2

$$\therefore \quad R = \infty$$

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

2003 – 2004

PAPER I(1) 2004

QUESTION

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SOLUTION 8(cont)

(iii)

$$f(x)=f(0)+f'(0)\cdot x+\frac{f''(0)}{2!}x^2+\dots$$

$f(x)=\ln(1+x)$

$f(0)=0$

$f'(x)=\frac{1}{1+x}$ $\quad f'(0)=1$ 1

$f''(x)=\frac{-1}{(1+x)^2}$ $\quad f''(0)=-1$ 1

$f'''(x)=\frac{2}{(1+x)^3}$ $\quad f'''(0)=2$ 1

$$\therefore F(x)=x-\frac{x^2}{2}+\frac{x^3}{3}\dots$$

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QUESTION

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SOLUTION Q9

a)

i) $(5+i12) = \sqrt{5^2+12^2}\,(\cos\theta + i\sin\theta)$, $\theta = \tan^{-1}\left(\frac{12}{5}\right)$

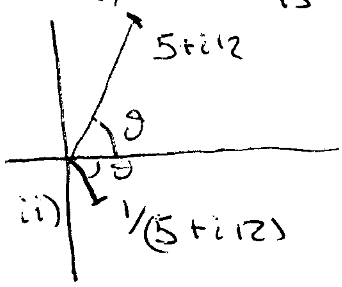
$= 13(\cos\theta + i\sin\theta) = 13e^{i\theta}$, $\tan\theta = \frac{12}{5}$ 1

ii) $1/(5+i12) = \dfrac{5-i12}{|5+i12|^2} = \dfrac{13(\cos(-\theta)+i\sin(-\theta))}{13^2}$

θ as above

$= \frac{1}{13}(\cos\theta - i\sin\theta)$

$= 1/13\ e^{-i\theta}$, $\tan\theta = 12/5$ 2



2

b) Have

(1) $\cos n\theta\cos\theta - \sin n\theta\sin\theta = x$, say

(2) $\cos n\theta\sin\theta + \sin n\theta\cos\theta = y$, say 2

Then (1) + i(2) $\Rightarrow$

$(\cos n\theta\cos\theta - \sin n\theta\sin\theta) + i(\cos n\theta\sin\theta + \sin n\theta\cos\theta) = x+iy$

LHS can be written

$(\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)$ 3

$= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta)$ de M's thm

$= (\cos\theta + i\sin\theta)^{n+1}$

$= \cos(n+1)\theta + i\sin(n+1)\theta$ de M's thm

$\Rightarrow x = \cos(n+1)\theta$, $y = \sin(n+1)\theta$ QED. 5

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Part I Paper 1 2004

Q10

Mark scheme

(i) $\cosh x = \dfrac{e^x + e^{-y}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\therefore \cosh x + \sinh x = e^x$

$\therefore (\cosh x + \sinh x)^n = e^{nx}$

$= \cosh nx + \sinh nx$ 4

(ii) $\cosh(x+iy) = \frac{1}{2}\left[e^{x+iy} + e^{-(x+iy)}\right]$

$= \frac{1}{2}\left[e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)\right]$

$= \cancel{\tfrac{1}{2}} \cosh x \cos y + i \sinh x \sin y$ — (*) 5

1

$\cosh^2 z = -1 \Rightarrow \cosh z = \pm i$

This matches (*) if ① $\sinh x = 1$, $\sin y = -1$, $\cos y = 0$

<u>or</u> ② $\sinh x = -1$, $\sin y = +1$, $\cos y = 0$

① $\Rightarrow y = (2n+1)\pi/2$ with $n = \pm 1, \pm 3, \ldots$ $x = \sinh^{-1} 1$ 5

② $\Rightarrow y = (2n+1)\pi/2$ with $n = 0, \pm 2, \ldots$ $x = -\sinh^{-1} 1$