UNIVERSITY OF LONDON [I(1) 2001]

B.ENG. AND M.ENG. EXAMINATIONS 2001

For Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examination for the Associateship.

PART I : MATHEMATICS 1

Wednesday 6th June 2001 10.00 am - 1.00 pm

Answer EIGHT questions.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

1. Let

$$f(x) \;=\; \frac{x+3}{2x+1}\,.$$

(i) Find the inverse function $f^{-1}(x)$ of $f(x)$.

(ii) Write $f(x)$ as the sum of an even and an odd function.

(iii) Find all solutions of the equation

$$f(f(x)) \;=\; 0\,.$$

(iv) Find all solutions of the equation

$$\frac{1}{f(\cos\theta)} \;=\; 0\,.$$

2. Consider the curve defined by the equation

$$y^2 \;=\; x^2 \;-\; \frac{x^4}{4}\,.$$

(i) Find the coordinates of all stationary points of the curve.

(ii) Find the coordinates of all points at which $\dfrac{dy}{dx}$ becomes infinite.

(iii) Sketch the curve.

PLEASE TURN OVER

3. Find $\dfrac{dy}{dx}$ in each of the following cases.

In case (v) *you may express your answer in terms of x and y.*

(i) $$y = e^{\sin x}.$$

(ii) $$y = \ln(\ln x).$$

(iii) $$y = x^2 e^x \cos x.$$

(iv) $$y = x^{\ln x}.$$

(v) $$xy + \ln(xy) = 1.$$

4. (i) Show that if $y = (\sin^{-1} x)^2$, then

$$(1 - x^2)^{1/2} \frac{dy}{dx} = 2 \sin^{-1} x.$$

Hence or otherwise show that y satisfies the equation

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 2 = 0.$$

(ii) Find the n^{th} derivatives of the functions $f(x) = e^{3x}$ and $h(x) = x^2 e^{3x}$.

(iii) Two sides of a triangle are of unit length and meet at angle θ. The length of the third side is given by $l(\theta) = (2 - 2\cos\theta)^{1/2}$. Find $dl/d\theta$.

By using the formula

$$\frac{dl}{d\theta} = \lim_{h \to 0} \frac{l(\theta + h) - l(\theta)}{h},$$

find the approximate change in l if θ changes from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{3} + 0.01$ (in radians).

5. Evaluate the following limits:

(i) $$\lim_{x \to 5} \frac{3 \; - \; \sqrt{x+4}}{x-5} \;;$$

(ii) $$\lim_{x \to 0} \; x^{-3} \; \tan^3(3x) \,;$$

(iii) $$\lim_{x \to 0} \frac{\ln(1+3x^2)}{1+x-e^x} \;;$$

(iv) $$\lim_{x \to \pi/3} \frac{1+\cos 3x}{\sqrt{3}-\tan x} \;.$$

6. Evaluate the following integrals :

(i) $$\int_1^e \frac{(\ln x)^2}{x} \; dx \,;$$

(ii) $$\int_0^1 \sqrt{1-x^2} \; dx \;;$$

(iii) $$\int \frac{x \, dx}{(1+x^2)^2} \;;$$

(iv) $$\int \frac{x^2 \, dx}{(1+x^2)^2} \;.$$

PLEASE TURN OVER

7. (i) Express the function

$$\frac{2x}{(x^2+1)\,(x-1)}$$

in partial fraction form, and hence find

$$\int \frac{2x\,dx}{(x^2+1)\,(x-1)}\ .$$

(ii) Let

$$I_n \;=\; \int_0^{\pi} \sin^n x\,dx.$$

By integrating by parts, prove that for $n \geq 2$,

$$I_n \;=\; \frac{n-1}{n}\, I_{n-2}\,.$$

Hence find

$$\int_0^{\pi} \sin^6 x\,dx\,.$$

8. (i) Find which of the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+10}$.

(ii) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (n+1)\,x^n\,.$$

(iii) Find $\dfrac{d^n}{dx^n}\,(1-x)^{-2}$ and hence show that the Maclaurin expansion of $(1-x)^{-2}$ is given by the series in part (ii).

9. (i) Express each of the following in the form $a + ib$:

$$\text{(a) } (1+i)^2\,, \qquad \text{(b) } \frac{1+i}{1-i}\,, \qquad \text{(c) } \left(\frac{\sqrt{3}+i}{2}\right)^{101}.$$

(ii) Find all complex roots z of the equation

$$z^4 = \frac{1}{4}\,(1+i)^4\,.$$

Show on a diagram where these roots lie.

What is the sum of all the roots?

(iii) If $z = x + iy$, express the equation

$$z + \overline{z} = \frac{1}{z} + \frac{1}{\overline{z}}$$

in terms of x and y. Hence sketch the solution curves of this equation in the complex plane.

10. (i) (a) Define the functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ (where z is a complex number) in terms of the exponential function.

(b) Find all complex roots z of the equation $\tanh z = i$.

(c) Hence or otherwise find all roots of the equation $\tan^2(iz) = 1$.

(ii) If $z = x + iy$, find the real and imaginary parts of $\cos(z^2)$ in terms of trigonometric and hyperbolic functions of x and y.

Hence, find all complex numbers such that $\cos(z^2)$ is real.

END OF PAPER

PART I - MATHS I - 2001

EXAMINATION QUESTIONS/SOLUTIONS SESSION 2000-2001

COURSE: B. Eng.

Setters are advised that Checkers, Editors, Typists and External Examiners greatly appreciate the merits of accuracy, legibility and neatness.

Write on one side only, between the margins, double-spaced. Not more than one question or solution per sheet, please

SETTER: CROWDY

QUESTION NO.

SOLUTION NO. 1

Mark Scheme

(a) $f = \dfrac{x+3}{2x+1} \Rightarrow (2x+1)f = x+3$

$\Rightarrow (2f-1)x = 3-f$

$\Rightarrow x(f) = \dfrac{3-f}{2f-1}$

$\therefore f^{-1}(x) = \dfrac{3-x}{2x-1}$ — 3

(b) $f(x) = \left[\dfrac{f(x)+f(-x)}{2}\right] + \left[\dfrac{f(x)-f(-x)}{2}\right]$

$= \underbrace{\dfrac{2x^2-3}{4x^2-1}}_{\text{even}} + \underbrace{\dfrac{5x}{4x^2-1}}_{\text{odd}}$ — 3

(c) $f(f(x)) = \dfrac{f(x)+3}{2f(x)+1} = \dfrac{\dfrac{x+3}{2x+1}+3}{\dfrac{2(x+3)}{2x+1}+1}$

$= \dfrac{7x+6}{4x+7}$ — 5

$\therefore f(f(x)) = 0 \Longleftrightarrow x = -\dfrac{6}{7}$

(d) $\dfrac{1}{f(\cos\theta)} = \dfrac{2\cos\theta+1}{\cos\theta+3} = 0 \Longleftrightarrow \cos\theta = -\tfrac{1}{2}$ — 4

$\therefore \theta = \begin{cases} 2\pi/3 + 2k\pi \\ 4\pi/3 + 2k\pi \end{cases}$ k any integer

SETTER: D. CROWDY

SETTER'S SIGNATURE: D. Crowdy

CHECKER: WILSON

CHECKER'S SIGNATURE: J. Wilson

DATE: 7/12/00

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 – 2001

Please write on this side only, legibly and neatly, between the margins

PAPER 1

QUESTION

SOLUTION 2

(a) $y^2 = x^2 - x^4/4$

Differentiating wrt x:

$$2y\frac{dy}{dx} = 2x - x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x - x^3}{2y} = \frac{x(2-x^2)}{2y} \quad (A)$$

Stat. pts $\Leftrightarrow$ $\frac{dy}{dx} = 0$

Only stat pts are at $x = \pm\sqrt{2}$

i.e $(\sqrt{2}, 1)$; $(\sqrt{2}, -1)$; $(-\sqrt{2}, 1)$; $(-\sqrt{2}, -1)$ 6

N.B no stat pt at $(0,0)$ – look at limit as $x \to 0$

(b) From (A), $\frac{dy}{dx} \to \infty$ when $\frac{dy}{dx} = \pm\frac{x(2-x^2)}{2x\sqrt{1-x^2/4}} \to \infty$ 4

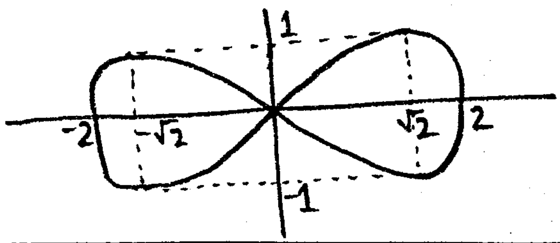
i.e when $x = \pm 2$

(c) Note curve invariant under transformations

$x \leftrightarrow -x$

$y \leftrightarrow -y$

$\Rightarrow$ reflectionally-symmetric in x & y axes 5



Setter : D. CROWDY

Checker : WILSON

Setter's signature : D Crowdy

Checker's signature : J. Wilson

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

Please write on this side only, legibly and neatly, between the margins

PAPER I.1

QUESTION

SOLUTION 3

i) $y' = \frac{d}{dx} e^u$, where $u = \sin x$,

$= \frac{du}{dx}\frac{d}{du} e^u = \cos x\, e^u = \cos x\, e^{\sin x}$. 3

ii) $y' = \frac{d}{dx} \ln u$, where $u = \ln x$,

$= \frac{du}{dx}\frac{d}{du}\ln u = \frac{1}{x}\frac{1}{u} = \frac{1}{x\ln x}$. 3

iii) $y' = (x^2 v)'$ where $v = e^x \cos x$

$= 2xv + x^2 v'$

$= 2x e^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x$,

(or use logarithmic differentiation). 3

iv) $\ln y = (\ln x)^2$.

$\therefore \frac{y'}{y} = \frac{2}{x}\ln x$, $y' = 2x^{-1} y \ln x = 2x^{\ln x - 1}\ln x$. 3

v) $xy + \ln x + \ln y = 1$,

$\therefore y + xy' + \frac{1}{x} + \frac{y'}{y} = 0$, $y'\frac{xy+1}{y} + \frac{xy+1}{x} = 0$. 3

$y' = -y/x$.

Setter : C J RIDLER-ROWE Setter's signature : C J Ridler-Rowe

Checker : J. R. CASH Checker's signature : J R Cash

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

Please write on this side only, legibly and neatly, between the margins

PAPER I.1

QUESTION

SOLUTION 4

i) $y' = 2(1-x^2)^{-1/2}\sin^{-1}x$ so $(1-x^2)^{1/2}y' = 2\sin^{-1}x.$ 2

$\therefore\ -2x\,\frac{1}{2}(1-x^2)^{-1/2}y' + (1-x^2)^{1/2}y'' = 2(1-x^2)^{-1/2},$

so $(1-x^2)y'' - xy' - 2 = 0.$ 3

ii) $y' = 3e^{3x},\ y'' = 3^2e^{3x},\ \ldots,\ y^{(n)} = 3^ne^{3x}.$ 2

With $f(x) = x^2$, $g(x) = e^{3x}$,

$h^{(n)} = (fg)^{(n)} = fg^{(n)} + {}^nC_1 f'g^{(n-1)} + {}^nC_2 f''g^{(n-2)} + \ldots$

$= x^2 3^n e^{3x} + n2x3^{n-1}e^{3x} + n(n-1)3^{n-2}e^{3x}.$ 2

iii) $\frac{df}{d\theta} = \sin\theta\,(2-2\cos\theta)^{-1/2}.$ 2

$f(\theta+h) - f(\theta) \simeq h\frac{df}{d\theta}$ for h small. 2

$\therefore$ For $\theta = \pi/3$, $h = 0.01$,

the change in f is $\simeq 0.01 . \frac{\sqrt{3}}{2} . 1 = \frac{1.732}{200}$

$= 0.00866.$ 2

Setter : RIDLER-ROWE Setter's signature : [signature]

Checker : J. R. CASH Checker's signature : JRCash

PAPER I (

QUESTION

SOLUTION 5

(i) $$\frac{3-\sqrt{x+4}}{x-5} = \frac{5-x}{(x-5)(3+\sqrt{x+4})} = \frac{-1}{3+\sqrt{x+4}}.$$

As $x \to 5$, this $\to \dfrac{-1}{3+\sqrt{9}} = -\frac{1}{6}$ 3

(ii) $$x^{-3}\tan^3(3x) = \left(\frac{\sin 3x}{3x}\right)^3 \cdot \frac{27}{\cos^3(3x)}.$$

As $x \to 0$, this $\to 1^3 \cdot \dfrac{27}{1^3} = 27$ 3

(iii) $$\frac{\ln(1+3x^2)}{1+x-e^x} = \frac{+3x^2+\dots}{-x^2/2+\dots}.$$

As $x \to 0$, this $\to \dfrac{3}{-1/2} = -6$. 4

(iv) By l'Hôpital's rule,

$$\lim_{x\to\pi/3} \frac{1+\cos 3x}{\sqrt{3}-\tan x} = \lim_{x\to\pi/3} \frac{-3\sin 3x}{-\sec^2 x}$$ 5

$$= \lim_{x\to\pi/3} (+3\sin 3x \cos^2 x) = 0$$

Setter : WILSON

Checker : HALL

Setter's signature : J. Wilson

Checker's signature : [signature]

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

Please write on this side only, legibly and neatly, between the margins

PAPER I (i)

QUESTION

SOLUTION 6

(i) Set $u = \ln x$, so $du = dx/x$. The integral becomes $\int_0^1 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$. 3

(ii) Set $x = \sin u$. The integral becomes

$$\int_0^{\pi/2} \cos^2 u\,du = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2u)\,du$$ 4

$$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\pi/2} = \pi/4.$$

(iii) Set $x^2 = u$. The integral becomes

$$\int \frac{\frac{1}{2}\,du}{(1+u)^2} = \frac{-1}{2(1+u)} = -\frac{1}{2(1+x^2)}\ (+c).$$ 3

(iv) By (iii), the integrand is $x\frac{d}{dx}\left(-\frac{1}{2(1+x^2)}\right)$.

Integrating by parts, we get that the given integral is $= -\frac{x}{2(1+x^2)} + \frac{1}{2}\int\frac{dx}{1+x^2}$ 5

$$= +\frac{1}{2}\left[\tan^{-1}x - \frac{x}{1+x^2}\right]\ (+c).$$

Setter : WILSON Setter's signature : J. Wilson

Checker : HALL Checker's signature : PHall

PAPER I(i)
QUESTION
SOLUTION 7

(i) $$\frac{2x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}\ ; \text{ so}$$

$$2x = (Ax+B)(x-1) + C(x^2+1).$$

Comparing coefficients $\Rightarrow A = -1,\ B = C = 1$, so

$$\frac{2x}{(x^2+1)(x-1)} = \frac{-x+1}{x^2+1} + \frac{1}{x-1}$$ 5

$$\therefore \int \frac{2x\,dx}{(x^2+1)(x-1)} = -\tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + \ln|x-1|,\ (+c).$$ 3

(ii) $$I_n = \int_0^\pi \sin^n x\,dx = -\int_0^\pi \sin^{n-1}x \frac{d}{dx}(\cos x)\,dx$$

$$= -\left[\sin^{n-1}x \cos x\right]_0^\pi + \int_0^\pi (n-1)\sin^{n-2}x \cos^2 x\,dx.$$ 5

$$= 0 + (n-1)\int_0^\pi \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\left(I_{n-2} - I_n\right).$$

Hence $nI_n = (n-1)I_{n-2}$, i.e. $I_n = \frac{n-1}{n}I_{n-2}$

$$\therefore I_6 = \frac{5}{6}I_4 = \frac{5}{6}\frac{3}{4}I_2 = \frac{5}{6}\frac{3}{4}\frac{1}{2}I_0$$ 2

$$= \frac{5}{16}I_0 = \frac{5\pi}{16}.$$

Setter : WILSON Setter's signature : J. Wilson
Checker : HALL Checker's signature : P Hall

PAPER I . 1

QUESTION

SOLUTION 8

i) a) Using the Ratio Test

$$\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \frac{n+1}{n}\frac{1}{2} \to \frac{1}{2} \text{ as } n \to \infty.$$

Limit is < 1. $\therefore$ Series converges. 2

b) Using the Ratio Test

$$\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!} = \frac{n+1}{2} \to \infty \text{ as } n \to \infty.$$

Limit is > 1 $\therefore$ Series diverges. 2

c) n^{th} term $\to 1$ as $n \to \infty$. $\therefore$ Divergent since n^{th} term $\not\to 0$. 2

ii) Fix $x \neq 0$. $\left|\frac{(n+1)^{th}\text{ term}}{n^{th}\text{ term}}\right| = \frac{n+2}{n+1}|x| \to |x|$ as $n \to \infty$.

$\therefore$ By Ratio Test, the series converges if limit $|x| < 1$
and diverges if limit $|x| > 1$.

So radius of convergence $= 1$. 4

iii) Put $f(x) = (1-x)^{-2}$. $f' = 2(1-x)^{-3}$, $f'' = 2.3(1-x)^{-4}$,
$f''' = 2.3.4\,(1-x)^{-5}, \ldots, f^{(n)} = (n+1)!\,(1-x)^{-n-2}$. 3

Maclaurin series has n^{th} term

$$\frac{f^{(n)}(0)}{n!}x^n = \frac{(n+1)!}{n!}x^n = (n+1)\,x^n$$

giving the series in ii). 2

Setter : C. J. RIDLER-ROWE Setter's signature : C J Ridler Rowe
Checker : J. R. CASH Checker's signature : JR Cash

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

Please write on this side only, legibly and neatly, between the margins

PAPER 1

QUESTION

SOLUTION 9

(i) (a) $(1+i)^2 = 2i$ 1

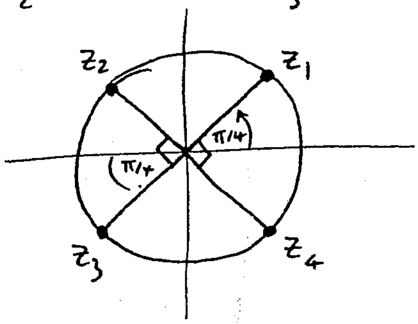
(b) $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = i$ 2

(c) $\left(\frac{\sqrt{3}+i}{2}\right)^{101} = \left(e^{i\pi/6}\right)^{101} = e^{16i\pi + \frac{5i\pi}{6}} = \frac{(-\sqrt{3}+i)}{2}$ 2

(ii) $z^4 = \frac{1}{4}(1+i)^4 = \left(\frac{1+i}{\sqrt{2}}\right)^4 = \left(e^{i\pi/4}\right)^4 = e^{i\pi + 2in\pi}$

$\Rightarrow \quad z = e^{i\pi/4 + in\pi/2} \qquad n = 0, 1, 2, 3$

$z_1 = e^{i\pi/4}, \quad z_2 = e^{3i\pi/4}, \quad z_3 = e^{5i\pi/4}, \quad z_4 = e^{7i\pi/4}$ 3



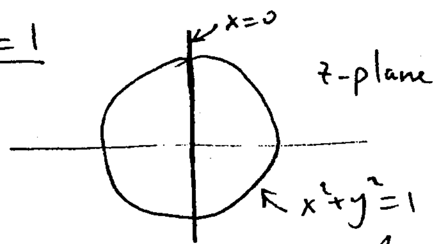
2

$\sum z_i = 0$ from the diagram 1

(iii) $x + iy + x - iy = \frac{1}{x+iy} + \frac{1}{x-iy}$

$2x = \frac{x-iy}{x^2+y^2} + \frac{x+iy}{x^2+y^2} \Rightarrow 2x = \frac{2x}{x^2+y^2}$

$\underline{x = 0}$ OR $\underline{x^2+y^2=1}$ 3



1

Setter : D. T. PAPAGEORGIOU Setter's signature :

Checker : R. T. FENNER Checker's signature : R. T. Fenner

MATHEMATICS FOR ENGINEERING STUDENTS

EXAMINATION QUESTION / SOLUTION

SESSION : 2000 - 2001

Please write on this side only, legibly and neatly, between the margins

PAPER	QUESTION	SOLUTION
1		10

(i) (a) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ 2

$\sinh z = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

(b) $\dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = i \Rightarrow e^{2z} = \dfrac{1+i}{1-i} = i$

$|i| = 1$ $\arg(i) = \frac{\pi}{2} + 2n\pi$ $\Rightarrow$ $2z = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)$

$z = i\left(\frac{\pi}{4} + n\pi\right)$ any integer n. 3

(c) $\tan^2 iz = 1 \Rightarrow \tan iz = 1 \Rightarrow i\tanh z = 1 \Rightarrow \tanh z = i$

$\tan iz = -1 \Rightarrow i\tanh z = -1 \Rightarrow \tanh z = -i$ 2

$\tanh z = i$ as in part (b)

$\tanh z = -i \Rightarrow \tanh(-z) = i \Rightarrow -z = i\left(\frac{\pi}{4} + n\pi\right)$

$z = i\left(-\frac{\pi}{4} + n\pi\right)$ 2

n any integer

(ii) $\cos(z^2) = \cos(x^2 - y^2 + 2ixy) = \cos(x^2 - y^2)\cos(2ixy)$

$- \sin(x^2 - y^2)\sin(2ixy)$

$= \cos(x^2 - y^2)\cosh 2xy - i\sin(x^2 - y^2)\sinh 2xy$ 3

$\cos(z^2)$ real $\Rightarrow \sin(x^2 - y^2)\sinh 2xy = 0$ 3

$\Rightarrow \underline{x = 0}$ or $\underline{y = 0}$ or $\underline{x^2 - y^2 = n\pi}$

n any integer

Setter : D. T. PAPAGEORGIOU Setter's signature :

Checker : R. T. FENNER Checker's signature :