MA2604

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

Part 2

Calculus and Vector Calculus

2013

Time allowed: 3 hours

Full marks may be obtained for correct answers to FIVE out of the SIX questions. All necessary working must be shown.

Important note for students regarding past exam papers
Past exam papers are published for illustrative purposes only. They can be used as a study aid but do not provide a definitive guide to either the format of the next exam, the topics that will be examined or the style of questions that will be set. Students should not expect their own exam to be directly comparable with previous papers. Remember that a degree requires an amount of self-study, reading around topics, and lateral thinking particularly at the higher level modules and for higher marks. Specific guidance for your exam will be given by the lecturer.

Each question carries 20 marks

Part A: Calculus

1. (a) Let $f(x, y) = x^2 \cos y$, with $x = x(t)$ and $y = y(t)$. Use the chain rule to express
   (i) $\frac{df}{dt}$ in terms of the derivatives of $x(t)$ and $y(t)$. [**4**]
   (ii) $\frac{d^2 f}{dt^2}$ in terms of the derivatives of $x(t)$ and $y(t)$. [**6**]

   (b) Consider the coordinate transformation

$$x = s \cos t, \quad y = \sin(s - t).$$

   (i) Write the Jacobian matrix of the transformation. [**4**]
   (ii) Now let $f(x, y) = x^2 - xy$ and calculate $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$. [**6**]

2. Consider the function

$$f(x, y) = xy - \log(x^2 + y^2).$$

   (i) Find the stationary points of f. [**6**]
   (ii) Using the second-partials test determine the nature of the stationary points. [**8**]
   (iii) Write down the Taylor expansion of f around each of the stationary points up to and including terms of second order. [**6**]

3. Consider the following differential equation ($x > 0$):

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = \frac{1}{\sqrt{x}}. \tag{1}$$

(i) Check that the function

$$y_1(x) = \frac{\cos x}{\sqrt{x}}$$

is a solution of the homogeneous differential equation associated to Eq. (1). [**4**]

(ii) Find the other independent solution $y_2(x)$ of the homogeneous equation. [**4**]

(iii) Calculate the Wronskian of the two solutions y_1 and y_2 and show that it is indeed different from zero. [**4**]

(iv) Using the method of variation of parameters, calculate a particular solution $y_p(x)$ of Eq (1). [**6**]

(v) Write down the general solution of Eq. (1). [**2**]

Part B: Vector Calculus

4. (a) Give a parametrization of the interior of the torus with Cartesian equation $y^2 + (3 - \sqrt{x^2 + z^2})^2 = 4$. [**10**]

   (b) Use your parametrization to find the volume V that lies inside the torus. [**10**]

5. Consider the ellipsoid $\mathcal{E}$ with Cartesian equation in standard form

$$\frac{x^2}{4} + \frac{y^2}{12} + \frac{z^2}{5} = 1\,.$$

   (a) Find all the points on $\mathcal{E}$ where the tangent plane is orthogonal to the plane of equation $z = 0$. The solution is a curve C whose Cartesian equations you should give. [**10**]

   (b) Find the line integral of the following vector field along C:

$$\vec{V}(x,y,z) = -y\vec{i} + x\vec{j}\,.$$

[**10**]

6. (a) Let $\vec{V}(x,y,z) = y(z-2)\vec{i} + z(x-2)\vec{j} + x(y-2)\vec{k}$. Find $\vec{\nabla} \times \vec{V}$. [**4**]

   (b) State Stokes theorem. [**2**]

   (c) Compute $\int_\Gamma \vec{V} \cdot d\vec{r}$ where Γ is the ellipse with equation

$$\begin{cases} \frac{(x+2)^2}{16} + \frac{(y-3)^2}{9} = 1\,, \\ z = -2\,. \end{cases}$$

   (You may use the formula for the area of an ellipse without justification.) [**14**]

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